SUPPLEMENTARY MATERIAL FOR "THEORY OF FUNCTIONAL PRINCIPAL COMPONENT ANALYSIS FOR DISCRETELY OBSERVED DATA"

BY HANG ZHOU[1], DONGYI WEI[2] AND FANG YAO[3]

[1]Department of Statistics, University of California at Davis, hgzhou@ucdavis.edu

[2]School of Mathematical Sciences, Peking University, jnwdyi@pku.edu.cn

[3]School of Mathematical Sciences and Center for Statistical Science, Peking University, fyao@math.pku.edu.cn

This supplement contains detailed proofs of the theorems presented in Sections 4 and 5 of the main text, along with additional corollaries and supporting lemmas.

S1. Proofs of theorems in Section 4

PROOF OF THEOREM 2. By equation (5.22) in Hsing and Eubank (2015), $\hat{\lambda}_j - \lambda_j$ admits the following expansion,

$$\hat{\lambda}_j-\lambda_j=\iint\{\hat{C}(s,t)-C(s,t)\}\phi_j(s)\phi_j(t)+\langle(\hat{\mathcal{P}}_j-\mathcal{P}_j)(\hat{C}-\hat{\lambda}_j\mathcal{I})(\hat{\mathcal{P}}_j-\mathcal{P}_j)\phi_j,\phi_j\rangle, \tag{S.1}$$

where $\hat{\mathcal{P}}_j=\hat{\phi}_j\otimes\hat{\phi}_j, \mathcal{P}_j=\phi_j\otimes\phi_j$ and $\mathcal{I}$ is the identity operator on $\mathcal{L}^2[0,1]$. For an operator $\mathcal{A}$ on $\mathcal{L}^2[0,1]$, we use $\|\mathcal{A}\|$ denotes its operator norm. By Lemma 5.1.7 in Hsing and Eubank (2015) and Taylor expansion of $\sqrt{1-x}$ at 0,

$$\begin{aligned}\|\hat{\phi}_j-\phi_j\|^2 =&2\{1-(1-\|\hat{\mathcal{P}}_j-\mathcal{P}_j\|^2)^{1/2}\}\\ =&2\left\{1-1+\frac{\|\hat{\mathcal{P}}_j-\mathcal{P}_j\|^2}{2}+o(\|\hat{\mathcal{P}}_j-\mathcal{P}_j\|^2)\right\}\\ =&\|\hat{\mathcal{P}}_j-\mathcal{P}_j\|^2+o(\|\hat{\mathcal{P}}_j-\mathcal{P}_j\|^2).\end{aligned} \tag{S.2}$$

Combine (S.1), (S.2) and Cauchy–Schwarz inequality,

$$\hat{\lambda}_j-\lambda_j=\iint\{\hat{C}(s,t)-C(s,t)\}\phi_j(s)\phi_j(t)+O(\|\hat{\phi}_j-\phi_j\|^2)+o(\|\hat{\phi}_j-\phi_j\|^2). \tag{S.3}$$

We first focus on the asymptotic behavior of $\iint\{\hat{C}(s,t)-C(s,t)\}\phi_j(s)\phi_j(t)$. Note that

$$\begin{aligned}&\iint\{\hat{C}(s,t)-C(s,t)\}\phi_j(s)\phi_j(t)\\ =&\iint\tilde{C}_0(s,t)\phi_j(s)\phi_j(t)+\iint\{\hat{C}(s,t)-C(s,t)-\tilde{C}_0(s,t)\}\phi_j(s)\phi_j(t),\end{aligned} \tag{S.4}$$

where

$$\tilde{C}_0(s,t)=\left\{R_{00}-C(s,t)S_{00}-h\frac{\partial C(s,t)}{\partial s}(s,t)S_{10}-h\frac{\partial C(s,t)}{\partial t}(s,t)S_{01}\right\}/f(s)f(t).$$

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For the bias term of $\iint \tilde{C}_0(s,t)\phi_j(s)\phi_j(t)$, by similar analysis of equation (S.30) in the proof of Lemma 1, under $hj^a \log n \lesssim 1$,

$$\mathbb{E}\left\{\int \tilde{C}_0(s,t)\phi_j(s)\phi_j(t)\right\} = h^2 \mathrm{K}_2 \lambda_j \int \phi_j^{(2)}\phi_j + o(h^2 j^{-a}). \tag{S.5}$$

For the variance of $\int \tilde{C}_0(s,t)\phi_j(s)\phi_j(t)$, denote

$$\beta_1 = \iint R_{00}(s,t)\frac{\phi_j(s)\phi_j(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t; \quad \beta_2 = \iint S_{00}(s,t)C(s,t)\frac{\phi_j(s)\phi_j(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t;$$

$$\beta_3 = h\iint \frac{\partial C(s,t)}{\partial s}S_{10}(s,t)\frac{\phi_j(s)\phi_j(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t; \quad \beta_3 = h\iint \frac{\partial C(s,t)}{\partial t}S_{01}(s,t)\frac{\phi_j(s)\phi_j(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t.$$

Then,

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left(\iint \tilde{C}_0(s,t)\phi_j(s)\phi_j(t)\right) =& \mathbb{V}\mathrm{ar}(\beta_1) + \mathbb{V}\mathrm{ar}(\beta_2) + \mathbb{V}\mathrm{ar}(\beta_3) + \mathbb{V}\mathrm{ar}(\beta_4)\\ &- 2\,\mathrm{Cov}(\beta_1,\beta_2) - 2\,\mathrm{Cov}(\beta_1,\beta_3) - 2\,\mathrm{Cov}(\beta_1,\beta_4)\\ &+ 2\,\mathrm{Cov}(\beta_2,\beta_3) + 2\,\mathrm{Cov}(\beta_2,\beta_4) + 2\,\mathrm{Cov}(\beta_3,\beta_4).\end{aligned}$$

By similar analysis as in the proof of Theorem 1,

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\beta_1) =& a_n - \frac{\lambda_j^2}{n} + o(a_n);\\ \mathbb{V}\mathrm{ar}(\beta_2) =& \sum_{i=1}^n v_i^2\left\{4!\binom{N_i}{4}\lambda_j^2 + 4\binom{N_i}{2}\iint C(u,v)^2\frac{\phi_j^2(u)\phi_j^2(v)}{f(u)f(v)}\mathrm{d}u\mathrm{d}v\right\} - \frac{\lambda_j^2}{n} + o(a_n);\\ \mathrm{Cov}(\beta_1,\beta_2) =& \sum_{i=1}^n v_i^2\left\{4!\binom{N_i}{4}\lambda_j^2 + 4\binom{N_i}{2}\iint C(u,v)^2\frac{\phi_j^2(u)\phi_j^2(v)}{f(u)f(v)}\mathrm{d}u\mathrm{d}v\right\} - \frac{\lambda_j^2}{n} + o(a_n);\\ \mathbb{V}\mathrm{ar}(\beta_2) =& o(a_n); \quad \mathrm{Cov}(\beta_1,\beta_3) = o(a_n); \quad \mathrm{Cov}(\beta_1,\beta_4) = o(a_n);\\ \mathrm{Cov}(\beta_2,\beta_3) =& o(a_n); \quad \mathrm{Cov}(\beta_2,\beta_4) = o(a_n); \quad \mathrm{Cov}(\beta_3,\beta_4) = o(a_n).\end{aligned}$$

where

$$\begin{aligned}a_n =& \sum_{i=1}^n v_i^2\left\{4!\binom{N_i}{4}\mathbb{E}(\xi_j^4) + 4!\binom{N_i}{3}\lambda_j\int\{C(u,u)+\sigma_X^2\}\frac{\phi_j^2(u)}{f(u)}\mathrm{d}u\right.\\ &\left.+4\binom{N_i}{2}\left[\int\{C(u,u)+\sigma_X^2\}\frac{\phi_j^2(u)}{f(u)}\mathrm{d}u\right]^2\right\}.\end{aligned}$$

Combine all above, we get $\mathbb{V}\mathrm{ar}\left(\iint \tilde{C}_0(s,t)\phi_j(s)\phi_j(t)\right) = \Sigma_n + o(\Sigma_n)$ with

$$\begin{aligned}\Sigma_n =& \frac{4!P_0(N)}{n}\frac{\mathbb{E}(\xi_j^4)-\lambda_j^2}{\lambda_j^2} + 4!\frac{4!P_1(N)}{n}\frac{\lambda_j\int\{C(u,u)+\sigma_X^2\}\frac{\phi_j^2(u)}{f(u)}\mathrm{d}u}{\lambda_j^2}\\ &+4\frac{P_2(N)}{n}\left(\left[\int\{C(u,u)+\sigma_X^2\}\frac{\phi_j^2(u)}{f(u)}\mathrm{d}u\right]^2 - \iint C(u,v)^2\frac{\phi_j^2(u)\phi_j^2(v)}{f(u)f(v)}\mathrm{d}u\mathrm{d}v\right).\end{aligned} \tag{S.6}$$

The proof is completed by

$$\|\hat{C} - C - \tilde{C}_0\|_{\mathrm{HS}} = o_P(h^2 j^{-a}) + o_p(\Sigma_n^{1/2})$$

under $hj^{2a}\log n = o(1)$ and $\Sigma_n^{-1/2}\|\hat{\phi}_j - \phi_j\| = o_P(1)$ for all $j = o(n^{1/(2a+4)})$.

□

S2. Proofs of theorems in Section 5

PROOF OF THEOREM 3. For the first statement of Theorem 3, we focus on the case $p = 0,\ q = 0$ and the proofs for other cases are similar. Let $\chi(\rho) = \{n^{-\rho}(i,j) | i,j \in \mathbb{Z} \cap (0, n^{\rho})\}$ be an equally sized mesh on $[0,1]^2$ with grid size $n^{-\rho}$ by $n^{-\rho}$ for a positive ρ. Then,

$$\sup_{s,t\in[0,1]} |R_{00}(s,t) - \mathbb{E}R_{00}(s,t)| \le \sup_{(s,t)\in\chi(\rho)} |R_{00}(s,t) - \mathbb{E}R_{00}(s,t)| + D_1 + D_2 \tag{S.7}$$

where

$$D_1 = \sup_{\substack{(s_1,t_1),(s_2,t_2)\in[0,1]^2 \\ |s_1-s_2|\le n^{-\rho}, |t_1-t_2|\le n^{-\rho}}} |R_{00}(s_1,t_1) - R_{00}(s_2,t_2)|,$$

$$D_2 = \sup_{\substack{(s_1,t_1),(s_2,t_2)\in[0,1]^2 \\ |s_1-s_2|\le n^{-\rho}, |t_1-t_2|\le n^{-\rho}}} |\mathbb{E}R_{00}(s_1,t_1) - \mathbb{E}R_{00}(s_2,t_2)|.$$

Recall $\delta_{ijl} = X_{ij}X_{il} = \{X_i(t_{ij}) + \varepsilon_{ij}\}\{X_i(t_{il}) + \varepsilon_{il}\}$ and for all $s_1, s_2, t_1, t_2 \in [0,1]$

$$\begin{aligned}
&|R_{00}(s_1,t_1) - R_{00}(s_2,t_2)| \\
\le& \frac{1}{h^2}\sum_{i=1}^n v_i \sum_{1\le l_1\ne l_2\le N_i} |\delta_{il_1l_2}| \sup\left|\mathrm{K}\left(\frac{t_{il_1}-s_1}{h}\right) - \mathrm{K}\left(\frac{t_{il_1}-s_2}{h}\right)\right| \mathrm{K}\left(\frac{t_{il_2}-t_1}{h}\right) \\
&+ \frac{1}{h^2}\sum_{i=1}^n v_i \sum_{1\le l_1\ne l_2\le N_i} |\delta_{il_1l_2}| \sup\left|\mathrm{K}\left(\frac{t_{il_1}-t_1}{h}\right) - \mathrm{K}\left(\frac{t_{il_1}-t_2}{h}\right)\right| \mathrm{K}\left(\frac{t_{il_2}-s_2}{h}\right) \\
\lesssim& \frac{1}{h^2}\sum_{i=1}^n v_i \sum_{1\le l_1\ne l_2\le N_i} |\delta_{il_1l_2}| \sup \frac{|s_1-s_2| + |t_1-t_2|}{h} \le Z_1 \frac{2n^{-\rho}}{h^3},
\end{aligned}$$

where

$$Z_1 := \sum_{i=1}^n v_i \sum_{1\le l_1\ne l_2\le N_i} |\delta_{il_1l_2}| = O(1) \text{ a.s.}$$

For the first term in the right hand side of equation (S.7), in order to use the Bernstein inequality, we need to do truncation on $\delta_{il_1l_2}$. Denote

$$\tilde{R}_{00}(s,t) = \sum_{i=1}^n v_i \sum_{1\le l_1\ne l_2\le N_i} \frac{1}{h^2}\mathrm{K}\left(\frac{t_{il_1}-s}{h}\right)\mathrm{K}\left(\frac{t_{il_2}-t}{h}\right)\delta_{il_1l_2}\mathbf{1}_{(|\delta_{il_1l_2}|\le A_n)}$$

where A_n is a positive constant we will define later. Then

$$\sup_{(s,t)\in\chi(\rho)} |R_{00}(s,t) - \mathbb{E}R_{00}(s,t)| \le \sup_{(s,t)\in\chi(\rho)} |\tilde{R}_{00}(s,t) - \mathbb{E}\tilde{R}_{00}(s,t)| + E_1 + E_2 \tag{S.8}$$

with

$$E_1 = \sup_{(s,t)\in\chi(\rho)} \sum_{i=1}^n v_i \sum_{1\le l_1\ne l_2\le N_i} \frac{1}{h^2}\mathrm{K}\left(\frac{t_{il_1}-s}{h}\right)\mathrm{K}\left(\frac{t_{il_2}-t}{h}\right)|\delta_{il_1l_2}|\mathbf{1}_{(|\delta_{il_1l_2}|> A_n)},$$

$$E_2 = \sup_{(s,t)\in\chi(\rho)} \mathbb{E}\sum_{i=1}^{n} v_i \sum_{1\le l_1\neq l_2\le N_i} \frac{1}{h^2}\mathrm{K}\left(\frac{t_{il_1}-s}{h}\right)\mathrm{K}\left(\frac{t_{il_2}-t}{h}\right)|\delta_{il_1l_2}|\mathbf{1}_{(|\delta_{il_1l_2}|>A_n)}.$$

Note that $\mathbb{E}|\delta_{il_1l_2}|^\alpha \lesssim 1$, $|E_1+E_2| = O(A_n^{1-\alpha}h^{-2})$ a.s. under Assumption 5, since

$$\sup_{s,t}|E_1+E_2| \lesssim \frac{1}{n}\sum_{i=1}^{n}\frac{1}{N_i(N_i-1)}\frac{1}{h^2}\sum_{1\le l_1\neq l_2\le N_i}\sup_{s,t}|\delta_{il_1l_2}|\mathbf{1}_{(|\delta_{il_1l_2}|>A_n)}$$

$$\lesssim \frac{1}{h^2A_n^{\alpha-1}} \lesssim A_n^{1-\alpha}h^{-2}\sum_{i=1}^{n}v_i\sum_{1\le l_1\neq l_2\le N_i}\sup_{s,t}|\delta_{il_1l_2}|^\alpha.$$

For each $(s,t)\in\chi(\rho)$, denote

$$L_i(s,t) = v_i\frac{1}{h^2}\sum_{1\le l_1\neq l_2\le N_i}\mathrm{K}\left(\frac{t_{il_1}-s}{h}\right)\mathrm{K}\left(\frac{t_{il_2}-t}{h}\right)\delta_{il_1l_2}\mathbf{1}_{(|\delta_{il_1l_2}|\le A_n)}.$$

Then, $\tilde{R}_{00}(s,t) = \sum_{i=1}^n L_i(s,t)$, $\mathbb{E}|L_i(s,t)|^2 \lesssim n^{-2}\{1+(N_ih)^{-2}\}$. Note that

$$|L_i(s,t)| \le A_n\frac{1}{n}\frac{1}{N_i(N_i-1)}\frac{1}{h^2}\sum_{1\le l_1\neq l_2\le N_i}\mathrm{K}\left(\frac{t_{il_1}-s}{h}\right)\mathrm{K}\left(\frac{t_{il_2}-t}{h}\right)$$

$$=A_n\frac{1}{n}\frac{N_i}{N_i-1}J_i(s)J_i(t),$$

where $J_i(s) := h^{-1}\sum_{l=1}^{N_i}\mathrm{K}((t_{il}-s)/h)$. For a sharp bound, we need a second truncation on J_i. Denote $Y_{il} = h^{-1}\mathrm{K}\left((t_{il}-s)/h\right)$. note that $\mathbb{E}Y_{il}=1$, $0\le Y_{il}\le 2/h$, for fixed $M\ge 5$, let $M' = M-1$ and by Bernstein inequality,

$$\mathbb{P}\left(J_i(s)>M\right) = \mathbb{P}\left(\frac{1}{N_i}\sum_{1\le l\le N_i}Y_{il}>M\right) = \mathbb{P}\left(\sum_{1\le l\le N_i}(Y_{il}-1)>M'N_i\right)$$

$$\le\exp\left(-\frac{(M'N_i)^2/2}{\sum_{l=1}^{N_i}\mathbb{E}|Y_{il}-1|^2+|2/h|M'N_i/3}\right)\le\exp\left(-\frac{(M'N_i)^2/2}{N_i|2/h|+|2/h|M'N_i/3}\right)$$

$$\le\exp\left(-\frac{(M'N_i)^2/2}{|2/h|M'N_i(1/4+1/3)}\right) = \exp(-3M'N_ih/7)\le\exp(-MN_ih/3).$$

Then

$$\mathbb{P}\left(|L_i(s,t)|>A_n\frac{1}{n}\frac{N_iM^2}{N_i-1}\right)\le\mathbb{P}\left(A_n\frac{1}{n}\frac{N_i}{N_i-1}J_i(s)J_i(t)>A_n\frac{1}{n}\frac{N_iM^2}{N_i-1}\right)$$

$$=\mathbb{P}\left(J_i(s)J_i(t)>M^2\right)\le\mathbb{P}\left(J_i(s)>M\right)+\mathbb{P}\left(J_i(t)>M\right)\le 2\exp(-MN_ih/3).$$

Denote

$$\widetilde{L}_i(s,t) = L_i(s,t)\mathbf{1}_{(|L_i(s,t)|\le A_n\frac{1}{n}\frac{N_iM^2}{N_i-1})} \text{ and } R^*_{00}(s,t) = \sum_{i=1}^{n}\widetilde{L}_i(s,t).$$

Then

$$\text{(S.9)}\qquad \sup_{(s,t)\in\chi(\rho)}|\tilde{R}_{00}(s,t)-\mathbb{E}\tilde{R}_{00}(s,t)|\le\sup_{(s,t)\in\chi(\rho)}|R^*_{00}(s,t)-\mathbb{E}R^*_{00}(s,t)|+F_1+F_2$$

with

$$F_1 = \sup_{(s,t)\in\chi(\rho)} \sum_{i=1}^n |L_i(s,t)| \mathbf{1}_{(|L_i(s,t)|>A_n \frac{1}{n}\frac{N_i M^2}{N_i-1})},$$

$$F_2 = \sup_{(s,t)\in\chi(\rho)} \mathbb{E} \sum_{i=1}^n |L_i(s,t)| \mathbf{1}_{(|L_i(s,t)|>A_n \frac{1}{n}\frac{N_i M^2}{N_i-1})}.$$

By similar arguments, $|F_1+F_2| = O(n^{2\rho}\left(1+\frac{1}{\bar{N}_2 h}\right)\exp(-M\bar{N}_2 h/6))$ a.s. by strong law of large number and

$$\mathbb{E}\left[|L_i(s,t)|\mathbf{1}_{(|L_i(s,t)|>A_n \frac{1}{n}\frac{N_i M^2}{N_i-1})}\right] \le \left[\mathbb{E}|L_i(s,t)|^2 \mathbb{P}\left(|L_i(s,t)| > A_n \frac{1}{n}\frac{N_i M^2}{N_i-1}\right)\right]^{1/2}$$
$$\le \frac{1}{n}\left(1+\frac{1}{\bar{N}_2 h}\right)\exp(-M\bar{N}_2 h/6).$$

Then $|\widetilde{L}_i(s,t) - \mathbb{E}\widetilde{L}_i(s,t)| \le 2A_n \frac{1}{n}\frac{N_i M^2}{N_i-1} \le 2A_n \frac{1}{n} M^2$, and

$$\sum_{i=1}^n \mathbb{E}\{\widetilde{L}_i(s,t) - \mathbb{E}\widetilde{L}_i(s,t)\}^2 \le \sum_{i=1}^n \mathbb{E}L_i^2(s,t) \lesssim \frac{1}{n}\left(1+\frac{1}{\bar{N}_2^2 h^2}\right).$$

By Bernstein inequality again, for $\lambda > 0$

$$\mathbb{P}\left(|R_{00}^*(s,t) - \mathbb{E}R_{00}^*(s,t)| > \lambda\right) = \mathbb{P}\left(\left|\sum_{i=1}^n (\widetilde{L}_i(s,t) - \mathbb{E}\widetilde{L}_i(s,t))\right| > \lambda\right)$$
$$\le 2\exp\left(-\frac{\lambda^2/2}{\sum_{i=1}^n \mathbb{E}\{\widetilde{L}_i(s,t) - \mathbb{E}\widetilde{L}_i(s,t)\}^2 + 2A_n \frac{1}{n} M^2\lambda/3}\right)$$
$$\lesssim 2\exp\left(-\frac{\lambda^2/2}{\frac{1}{n}\left(1+\frac{1}{\bar{N}_2^2 h^2}\right) + 2A_n \frac{1}{n} M^2 \lambda/3}\right).$$

Take $\lambda = \sqrt{\frac{(2\rho+2)\ln n}{n}}\left(1+\frac{1}{\bar{N}_2 h}\right) + A_n \frac{1}{n} M^2 (2\rho+2)\ln n$, then

$$\mathbb{P}\left(\sup_{(s,t)\in\chi(\rho)} |R_{00}^*(s,t) - \mathbb{E}R_{00}^*(s,t)| \ge \lambda\right)$$
$$\lesssim 2n^{2\rho}\exp\left(-\frac{\frac{(2\rho+2)\ln n}{n}\left(1+\frac{1}{\bar{N}_2^2 h^2}\right) + A_n^2 \frac{1}{n^2} M^4 (2\rho+2)^2 \ln^2 n/4}{\frac{1}{n}\left(1+\frac{1}{\bar{N}_2^2 h^2}\right) + 2A_n^2 \frac{1}{n^2} M^4 (2\rho+2)/3}\right)$$
$$\lesssim n^{-2}.$$

Then by Borel–Cantelli Lemma

$$\sup_{(s,t)\in\chi(\rho)} |R_{00}^*(s,t) - \mathbb{E}R_{00}^*(s,t)| = O\left(\sqrt{\frac{\rho \ln n}{n}}\left(1+\frac{1}{\bar{N}_2 h}\right) + A_n \frac{1}{n} M^2 \rho \ln n\right) \text{ a.s.}$$

Then

$$\sup_{s,t\in[0,1]}|R_{00}(s,t)-\mathbb{E}R_{00}(s,t)|\leq \sup_{(s,t)\in\chi(\rho)}|R_{00}^{*}(s,t)-\mathbb{E}R_{00}^{*}(s,t)|$$
$$+\mathbb{E}|D_1+D_2|+\mathbb{E}|E_1+E_2|+\mathbb{E}|F_1+F_2|$$
$$\lesssim\left[\sqrt{\frac{\rho\ln n}{n}}\left(1+\frac{1}{\bar{N}_2h}\right)+A_n\frac{1}{n}M^2\rho\ln n\right]+n^{-\rho}h^{-3}$$
$$+A_n^{1-\alpha}h^{-2}+n^{2\rho}\left(1+\frac{1}{\bar{N}_2^{1/2}h}\right)\exp(-M\bar{N}_2^{1/2}h/6).$$

The first statement of Theorem 3 is complete by taking $M=5+6(2\rho+1)\frac{\ln n}{\bar{N}_2^{1/2}h}$, $A_n=n^{1/\alpha}|M^2h^2\rho\ln n|^{-1/\alpha}$ and $\rho=3s+1$.

For the second statement of Theorem 3, by part (a) and similar arguments as Step 1 in the proof of Theorem 1, we get

$$\sup_{s,t\in[0,1]}|\hat{C}(s,t)-\mathbb{E}\hat{C}(s,t)|=O\left(\sqrt{\frac{\ln n}{n}}\left(1+\frac{1}{\bar{N}_2h}\right)+\left|\frac{\ln n}{n}\right|^{1-\frac{1}{\alpha}}\left|1+\frac{\ln n}{\bar{N}_2h}\right|^{2-\frac{2}{\alpha}}h^{-\frac{2}{\alpha}}\right).$$

Under Assumption 2, $|\mathbb{E}\hat{C}(s,t)-C(s,t)|=h^4$ for all $s,t\in[0,1]$, which complete the proof. □

PROOF OF THEOREM 4. In this proof, we will generalize the double truncation methodology in the proof of Theorem 3 to obtain the uniform convergence of eigenfunction. To concentrate on the key content of the proof and exclude non-essential calculations, we focus on the case where t_{ij} are uniform distributed and denote $\hat{C}_0=R_{00}(s,t)=n^{-1}\sum_{i=1}^n v_i\sum_{l_1\neq l_2}h^{-1}\mathrm{K}((t_{il_1}-s)/h)h^{-1}\mathrm{K}((t_{il_2}-t)/h)\delta_{il_1l_2}$ and assume $\{\hat{\phi}_k\}_{k=1}^{\infty}$ are eigenfunctions of $\hat{C}_0$. Let $\chi_1(\rho)=\{n^{\rho}\mathbf{1}_{(\frac{j}{n^\rho},\frac{j+1}{n^\rho})}|j\in\mathbb{Z}\cap[0,n^\rho)\}$, $(\rho\in\mathbb{Z}_+)$, note that

$$\sup_{s\in[0,1]}|\hat{\phi}_j(s)-\phi_j(s)|\leq\sup_{g\in\chi_1(\rho)}|\langle\hat{\phi}_j-\phi_j,g\rangle|+D_1+D_2 \tag{S.10}$$

where

$$D_1=\sup_{\substack{s_1,s_2\in[0,1]\\|s_1-s_2|\leq n^{-\rho}}}|\hat{\phi}_j(s_1)-\hat{\phi}_j(s_2)|,\quad D_2=\sup_{\substack{s_1,s_2\in[0,1]\\|s_1-s_2|\leq n^{-\rho}}}|\phi_j(s_1)-\phi_j(s_2)|.$$

As $\hat{\lambda}_j\hat{\phi}_j(s)=\int\hat{C}_0(s,t)\hat{\phi}_j(t)\mathrm{d}t$, and by the definition of $\hat{C}_0$ and Lipschitz continuous of the kernel function K, one has $|\hat{C}_0(s_1,t)-\hat{C}_0(s_2,t)|\lesssim Z_1|s_1-s_2|/h^3$, where

$$Z_1:=\sum_{i=1}^n v_i\sum_{1\leq l_1\neq l_2\leq N_i}|\delta_{il_1l_2}|\text{ and }\mathbb{E}Z_1=\mathbb{E}|\delta_{il_1l_2}|\lesssim 1.$$

Denote $\Omega_u:=\{\|\Delta\|_{\mathrm{HS}}\leq\eta_{j_{\max}}/2,\ \|\Delta\|_\infty j_{\max}^a\leq 1,\ \|\Delta\|_{\mathrm{HS}}j_{\max}^{a+1}\leq 1\}$ and $\mathbb{P}(\Omega_u)\to 1$ under assumptions of Theorem 4. Then on Ω_u

$$|\hat{\phi}_j(s_1)-\hat{\phi}_j(s_2)|\leq\hat{\lambda}_j^{-1}\int|\hat{C}_0(s_1,t)-\hat{C}_0(s_2,t)||\hat{\phi}_j(t)|\mathrm{d}t$$
$$\lesssim\hat{\lambda}_j^{-1}Z_1\frac{|s_1-s_2|}{h^3}\cdot\int|\hat{\phi}_j(t)|\mathrm{d}t\lesssim\hat{\lambda}_j^{-1}Z_1|s_1-s_2|/h^3$$
$$\lesssim j^aZ_1|s_1-s_2|/h^3,$$

where the last inequality holds on the high probability set Ω_u. Under Assumption 4, we also have $|\phi_j(s_1) - \phi_j(s_2)| \lesssim j^{c/2}|s_1 - s_2|$. Then we have

$$\mathbb{E}|D_1 + D_2| \lesssim (j^a h^{-3} + j^{c/2})n^{-\rho}.$$

By using the perturbation technique again, the following Lemma decompose $\hat{\phi}_j - \phi_j$ into two parts, where $\phi_{j,1}$ is the dominating term that requires subsequent analyses. Its proof can be found in Section S4.

LEMMA S1. *Under Assumptions 1 to 6, $h^4 j^{2a+2c} \lesssim 1$ and $hj^a \log n \lesssim 1$, there is*

$$\hat{\phi}_j - \phi_j = \phi_{j,0} + \hat{\lambda}_j^{-1}\phi_{j,1} \text{ with } \phi_{j,1} = \sum_{k=2j}^{\infty}\left\{\int (\hat{C}_0 - C)\phi_j\phi_k\right\}\phi_k$$

and

$$\begin{aligned}\|\phi_{j,0}\|_\infty =& O_p\Bigg(\frac{j\log j}{\sqrt{n}} + \frac{1}{\sqrt{n\bar{N}_2}}\left(j^{\frac{a}{2}+1}\log j + h^{-\frac{1}{2}}j^{\frac{a+1}{2}}\right) \\ &+\frac{j^{a+\frac{1}{2}}}{\sqrt{nh}\bar{N}_2} + \frac{1}{\sqrt{n}\bar{N}_2}j^{a+1}\log j + h^2 j^{c+1}\log j\Bigg)\end{aligned} \tag{S.11}$$

and $\phi_{j,1} = \sum_{k=2j}^{\infty}\langle\Delta\phi_j,\phi_k\rangle\phi_k$.

For fixed $g \in \chi_1(\rho)$ we have $\|g\|_1 = 1$ and $|\langle\hat{\phi}_j - \phi_j, g\rangle| = |\langle\phi_{j,0} + \hat{\lambda}_j^{-1}\phi_{j,1}, g\rangle| \le \|\phi_{j,0}\|_\infty + \hat{\lambda}_j^{-1}|\langle\phi_{j,1},g\rangle|$ by Lemma S1. For all $g \in \mathcal{L}^2$ denote $\mathcal{P}_{\ge m}g := \sum_{k=m}^{\infty}\langle g,\phi_k\rangle\phi_k$, $\mathcal{P}_{<m}g := \sum_{k=1}^{m-1}\langle g,\phi_k\rangle\phi_k$, and $g = \mathcal{P}_{<m}g + \mathcal{P}_{\ge m}g$. Thus,

$$\langle\phi_{j,1}, g\rangle = \sum_{k=2j}^{\infty}\left\{\int(\hat{C}_0 - C)\phi_j\phi_k\right\}\langle\phi_k, g\rangle = \int \hat{C}_0\phi_j\mathcal{P}_{\ge 2j}g.$$

The first term in equation (S.10) becomes

$$\sup_{g\in\chi_1(\rho)}|\langle\hat{\phi}_j - \phi_j, g\rangle| \le \|\phi_{j,0}\|_\infty + \hat{\lambda}_j^{-1}\sup_{g\in\chi_1(\rho)}\left|\int\hat{C}_0\phi_j\mathcal{P}_{\ge 2j}g\right| \tag{S.12}$$

and

$$\begin{aligned}&\sup_{g\in\chi_1(\rho)}\left|\int\hat{C}_0\phi_j\mathcal{P}_{\ge 2j}g\right| \\ \le& \sup_{g\in\chi_1(\rho)}\left|\int(\hat{C}_0 - \mathbb{E}\hat{C}_0)\phi_j\mathcal{P}_{\ge 2j}g\right| + \sup_{g\in\chi_1(\rho)}\left|\mathbb{E}\int\hat{C}_0\phi_j\mathcal{P}_{\ge 2j}g\right|.\end{aligned} \tag{S.13}$$

The following lemma bounds the bias term in the right hand side of (S.13) and its proof can be found in the Section S4.

LEMMA S2. *Under Assumptions 1 to 6, for all $g \in \chi_1(\rho)$,*

(i) $\|\mathcal{P}_{<m}g\|^2 \le m$, $\|\mathcal{P}_{<m}g\|_\infty \le m$ *and* $\|\mathcal{P}_{>m}g\|_1 \le m^{1/2}$.
(ii)

$$\sup_{g\in\chi_1(\rho)}\left|\mathbb{E}\int\hat{C}_0\phi_j\mathcal{P}_{\ge 2j}g\right| = O\left(h^2 j^{c+1-a}\right). \tag{S.14}$$

To bound the variance term, we shall extend the double truncation technique in Theorem 3 to the eigenfunction case. To be generic, for all $f, g \in \mathcal{L}^2$, recall that

$$\int \hat{C}_0 fg = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{N_i(N_i-1)} \sum_{l_1 \neq l_2} \delta_{il_1l_2} \mathcal{T}_h f(t_{il_1}) \mathcal{T}_h g(t_{il_2}),$$

where $\mathcal{T}_h f(x) = \frac{1}{h}\int \mathrm{K}\left(\frac{x-y}{h}\right) f(y)\mathrm{d}y$. Let $A_i(f,g) = \sum_{l_1\neq l_2} \delta_{il_1l_2}\mathcal{T}_h f(t_{il_1})\mathcal{T}_h g(t_{il_2})$ then

$$\int \hat{C}_0 fg = \frac{1}{n}\sum_{i=1}^{n} \frac{A_i(f,g)}{N_i(N_i-1)}, \quad \mathbb{E}\left(\int \hat{C}_0 fg\right) = \frac{\mathbb{E}A_i(f,g)}{N_i(N_i-1)}$$

Since $\delta_{il_1l_2}$ is unbounded random variable, we first truncate on $\delta_{il_1l_2}$. Denote

$$\tilde{C}(s,t) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{N_i(N_i-1)} \frac{1}{h^2} \sum_{1\le l_1 \neq l_2 \le N_i} \mathrm{K}\left(\frac{t_{il_1}-s}{h}\right)\mathrm{K}\left(\frac{t_{il_2}-t}{h}\right)\delta_{il_1l_2}\mathbf{1}_{(|\delta_{il_1l_2}|\le A_n)}$$

where A_n is a positive constant we will define later and then,

$$\text{(S.15)}\qquad \sup_{g\in\chi_1(\rho)}\left|\int(\hat{C}_0 - \mathbb{E}\hat{C}_0)\phi_j \mathcal{P}_{\ge 2j} g\right| \le \sup_{g\in\chi_1(\rho)}\left|\int(\tilde{C} - \mathbb{E}\tilde{C})\phi_j \mathcal{P}_{\ge 2j} g\right| + E_1 + \mathbb{E}E_1$$

with

$$E_1 = \sup_{g\in\chi_1(\rho)} \frac{1}{n}\sum_{i=1}^{n}\frac{1}{N_i(N_i-1)} \sum_{1\le l_1\neq l_2\le N_i} |\mathcal{T}_h\phi_j(t_{il_1})\mathcal{T}_h\mathcal{P}_{\ge 2j}g(t_{il_2})||\delta_{il_1l_2}|\mathbf{1}_{(|\delta_{il_1l_2}|>A_n)}.$$

For the first term in equation (S.15), let $\tilde{A}_i(f,g) = \sum_{l_1\neq l_2}\delta_{il_1l_2}\mathbf{1}_{(|\delta_{il_1l_2}|\le A_n)}\mathcal{T}_h f(t_{il_1})\mathcal{T}_h g(t_{il_2})$ and we obtain a trivial bound for $A_i(f,g)$,

$$\text{(S.16)}\qquad |\tilde{A}_i(f,g)| \le A_n N_i^2 J_i[\mathcal{T}_h f] J_i[\mathcal{T}_h g] \text{ with } J_i[\varphi] := \frac{1}{N_i}\sum_{l=1}^{N_i}|\varphi(t_{il})|,\ \forall\varphi\in\mathcal{L}^2[0,1].$$

However, the bound (S.16) is not optimal and to obtain a sharper bound this, we truncate on $\tilde{A}_i(f,g)$ with $A_n N_i^2\|f\|_\infty M$, where M is a positive number defined later. We start with the probability bound $\mathbb{P}(J_i[\varphi] > M)$ for all $\varphi\in\mathcal{L}^2[0,1]$, note that $\mathbb{E}|\varphi(t_{il})| = \|\varphi\|_1$, $|\varphi(t_{il})| \le \|\varphi\|_\infty$ and $\mathbb{E}|\varphi(t_{il})|^2 = \|\varphi\|^2$. For fixed $M \ge 4\|\varphi\|_1$ and $M' = M - 4\|\varphi\|_1$, by Bernstein inequality,

$$\mathbb{P}(J_i[\varphi] > M) = \mathbb{P}\left(\sum_{1\le l\le N_i}(|\varphi(t_{il})| - \|\varphi\|_1) > M'N\right)$$

$$\text{(S.17)}\qquad \begin{aligned} &\le \exp\left(-\frac{(M'N_i)^2/2}{\sum_{i=1}^{N_i}\mathbb{E}||\varphi(t_{il})| - \|\varphi\|_1|^2 + \|\varphi\|_\infty M'N_i/3}\right)\\ &\le \exp\left(-\frac{(M'N_i)^2/2}{N_i\|\varphi\|_2^2 + \|\varphi\|_\infty M'N_i/3}\right) \le \exp\left(-\frac{(M'N_i)^2/2}{\|\varphi\|_\infty(\|\varphi\|_1 N_i + M'N_i/3)}\right)\\ &\le \exp\left(-\frac{(M'N_i)^2/2}{\|\varphi\|_\infty(2M'N_i/3)}\right) \le \exp\left(-\frac{MN_i}{2\|\varphi\|_\infty}\right).\end{aligned}$$

Let $\tilde{A}_i^M(f,g) = \tilde{A}_i(f,g)\mathbf{1}_{(|\tilde{A}_i(f,g)|\le A_n N^2\|f\|_\infty M)}$. As $J_i[\mathcal{T}_h f] \le \|\mathcal{T}_h f\|_\infty \le \|f\|_\infty$, we have $|\tilde{A}_i(f,g)| \le A_n N_i^2 J_i[\mathcal{T}_h f]J_i[\mathcal{T}_h g] \le A_n N_i^2\|f\|_\infty J_i[\mathcal{T}_h g]$, thus

$$\mathbb{P}\left(|\tilde{A}_i(f,g)| > A_n N_i^2\|f\|_\infty M\right) \le \mathbb{P}(J_i[\mathcal{T}_h g] > M) \le \exp\left(-\frac{MN_i}{2\|\mathcal{T}_h g\|_\infty}\right).$$

Let

$$C_*^M(f,g) = \frac{1}{n}\sum_{i=1}^n \frac{\tilde{A}_i^M(f,g)}{N_i(N_i-1)} \text{ and } C_*^{>M}(f,g) = \frac{1}{n}\sum_{i=1}^n \tilde{A}_i(f,g)\mathbf{1}_{(|\tilde{A}_i(f,g)|>A_n N_i^2\|f\|_\infty M)},$$

we have $\int \tilde{C} fg = C_*^M(f,g) + C_*^{>M}(f,g)$ and

$$\int (\tilde{C} - \mathbb{E}\tilde{C})fg \le |C_*^M(f,g) - \mathbb{E}[C_*^M(f,g)]| + C_*^{>M}(f,g) + \mathbb{E}[C_*^{>M}(f,g)].$$

Denote

$$M_0 := \sup_{g\in\chi_1(\rho)} \|\mathcal{T}_h\mathcal{P}_{\le 2j}g\|_\infty, M_1 := \sup_{g\in\chi_1(\rho)} \|\mathcal{T}_h\mathcal{P}_{\le 2j}g\|_1, M_2 := \sup_{g\in\chi_1(\rho)} \mathbb{E}|\tilde{A}_i(\phi_j,\mathcal{P}_{\ge 2j}g)|^2.$$

Then for $M \ge 4M_1$, $M_1 := \sup_{g\in\chi_1(\rho)} \|\mathcal{T}_h\mathcal{P}_{\le 2j}g\|_1$,

$$\begin{aligned}&\sup_{g\in\chi_1(\rho)}\left|\int(\tilde{C}-\mathbb{E}\tilde{C})\phi_j\mathcal{P}_{\ge 2j}g\right| \\ &\le \sup_{g\in\chi_1(\rho)} |C_*^M(\phi_j,\mathcal{P}_{\ge 2j}g) - \mathbb{E}[C_*^M(\phi_j,\mathcal{P}_{\ge 2j}g)]| + F_1 + \mathbb{E}F_1\end{aligned} \tag{S.18}$$

with

$$F_1 = \sup_{g\in\chi_1(\rho)} C_*^{>M}(\phi_j,\mathcal{P}_{\ge 2j}g).$$

By Bernstein inequality, for all $t > 0$

$$\begin{aligned}&\mathbb{P}\left(|C_*^M(f,g) - \mathbb{E}[C_*^M(f,g)]| > t\right) \\ =&\mathbb{P}\left(\left|\sum_{i=1}^n v_i(\tilde{A}_i^M(f,g) - \mathbb{E}[\tilde{A}_i^M(f,g)])\right| > t\right) \\ \le& 2\exp\left(-\frac{n^2\bar{N}_2^2(\bar{N}_2-1)^2t^2/2}{\sum_{i=1}^n \mathbb{E}\{\tilde{A}_i^M(f,g) - \mathbb{E}[\tilde{A}_i^M(f,g)]\}^2 + 2A_n\bar{N}_2^2\|f\|_\infty Mn\bar{N}_2(\bar{N}_2-1)t/3}\right) \\ \le& 2\exp\left(-\frac{n^2\bar{N}_2^2(\bar{N}_2-1)^2t^2/2}{n\mathbb{E}|\tilde{A}_i^M(f,g)|^2 + 2A_n\bar{N}_2^2\|f\|_\infty Mn\bar{N}_2(\bar{N}_2-1)t/3}\right).\end{aligned}$$

We obtain the following expectation bound,

$$\begin{aligned}&\mathbb{E}\left[\sup_{g\in\chi_1(\rho)} |C_*^M(\phi_j,\mathcal{P}_{\ge 2j}g) - \mathbb{E}[C_*^M(\phi_j,\mathcal{P}_{\ge 2j}g)]|\right] \\ =&\int_0^\infty \left(\sup_{g\in\chi_1(\rho)} |C_*^M(\phi_j,\mathcal{P}_{\ge 2j}g) - \mathbb{E}[C_*^M(\phi_j,\mathcal{P}_{\ge 2j}g)]| > t\right) \mathrm{d}\,t \\ \le&\int_0^\infty \min\left\{1, 2n^\rho \exp\left(-\frac{n^2\bar{N}_2^2(\bar{N}_2-1)^2t^2/2}{nM_2 + 2A_n\bar{N}_2^2\|\phi_j\|_\infty Mn\bar{N}_2(\bar{N}_2-1)t/3}\right)\right\}\mathrm{d}\,t \\ \le&\text{Const}\left[\sqrt{\frac{\rho\ln n}{n}\frac{M_2}{\bar{N}_2^4}} + A_n\frac{M}{n}\rho\ln n\right].\end{aligned} \tag{S.19}$$

The following lemma bounds the truncated terms E_1 and F_1 in (S.15) and (S.18), and its proof can be found in Section S4.

LEMMA S3. *Under Assumptions 1 to 6, $h^4 j^{2a+2c} \lesssim 1$ and $hj^a \log n \lesssim 1$,*

$$\mathbb{E}|E_1| \lesssim \frac{1}{hA_n^{\alpha-1}} \text{ and } \mathbb{E}|F_1| \le \frac{n^\rho M_2^{1/2}}{\bar N_2(\bar N_2 - 1)} \exp\left(-\frac{M\bar N_2}{4M_0}\right)$$

Combine equation (S.15), (S.18) and Lemma S3,

$$\text{(S.20)}\quad \begin{aligned} &\sup_{g\in\chi_1(\rho)}\left|\int(\hat C_0 - \mathbb{E}\hat C_0)\phi_j \mathcal{P}_{\ge 2j} g\right| \\ \lesssim &\left[\sqrt{\frac{\rho \ln n}{n}\frac{M_2}{\bar N_2^4}} + A_n \frac{M}{n}\rho\ln n\right] + \frac{1}{hA_n^{\alpha-1}} + \frac{2n^\rho M_2^{1/2}}{\bar N_2(\bar N_2-1)}\exp\left(-\frac{M\bar N_2}{4M_0}\right). \end{aligned}$$

Recall that $M_2 = \sup_{g\in\chi_1(\rho)} \mathbb{E}|\tilde A_i(\phi_j, \mathcal{P}_{\ge 2j} g)|^2$, with

$$\tilde A_i(f,g) = \sum_{l_1\ne l_2} \delta_{il_1l_2}\mathbf{1}_{(|\delta_{il_1l_2}|\le A_n)}\mathcal{T}_h f(t_{il_1})\mathcal{T}_h g(t_{il_2}),$$

which implies that M_2 is depend on A_n. Consider the un-truncated version of M_2, $M_2^0 := \sup_{g\in\chi_1(\rho)} \mathbb{E}|A_i(\phi_j, \mathcal{P}_{\ge 2j} g)|^2$, where $A_i(f,g) = \sum_{l_1\ne l_2}\delta_{il_1l_2}\mathcal{T}_h f(t_{il_1})\mathcal{T}_h g(t_{il_2})$. We first study the difference between M_2 and M_2^0. Start with the difference between $A_i(f,g)$ and $\tilde A_i(f,g)$ for all $f, g\in\mathcal{L}^2$,

$$\text{(S.21)}\quad \begin{aligned} |A_i(f,g) - \tilde A_i(f,g)|^2 = &\left|\sum_{l_1\ne l_2}\delta_{il_1l_2}\mathbf{1}_{(|\delta_{il_1l_2}|> A_n)}\mathcal{T}_h f(t_{il_1})\mathcal{T}_h g(t_{il_2})\right|^2 \\ \le &\sum_{l_1\ne l_2}\delta_{il_1l_2}^2\mathbf{1}_{(|\delta_{il_1l_2}|> A_n)}\sum_{l_1\ne l_2}|\mathcal{T}_h f(t_{il_1})\mathcal{T}_h g(t_{il_2})|^2 \\ \le &A_n^{2-\alpha} Z_{2,i} N_i^4 J_i[|\mathcal{T}_h f|^2] J_i[|\mathcal{T}_h g|^2] \le A_n^{2-\alpha} Z_{2,i} N_i^4 \|f\|_\infty^2 \|\mathcal{T}_h g\|_\infty J_i[\mathcal{T}_h g], \end{aligned}$$

where

$$Z_{2,i} := \frac{1}{N_i(N_i-1)}\sum_{1\le l_1\ne l_2\le N_i}|\delta_{il_1l_2}|^\alpha \text{ with } \mathbb{E}Z_{2,i} = \mathbb{E}|\delta_{il_1l_2}|^\alpha \lesssim 1.$$

On the set $\{J_i[\mathcal{T}_h g]\le M\}$, we have

$$\text{(S.22)}\quad \begin{aligned} &\mathbb{E}\big[|A_i(f,g) - \tilde A_i(f,g)|^2\mathbf{1}_{(J_i[\mathcal{T}_h g]\le M)}\big] \le \mathbb{E}\big[A_n^{2-\alpha} Z_{2,i} N_i^4\|f\|_\infty^2\|\mathcal{T}_h g\|_\infty M\big] \\ &\le \text{Const} A_n^{2-\alpha} N_i^4 \|f\|_\infty^2 \|\mathcal{T}_h g\|_\infty M. \end{aligned}$$

On the set $\{J_i[\mathcal{T}_h g] > M\}$, under the condition $M\ge 4\|\mathcal{T}_h g\|_1$,

$$\text{(S.23)}\quad \begin{aligned} &\mathbb{E}\big[|A_i(f,g) - \tilde A_i(f,g)|^2\mathbf{1}_{(J_i[\mathcal{T}_h g]> M)}\big] \\ \le &\big|\mathbb{E}|A_i(f,g) - \tilde A_i(f,g)|^\alpha\big|^{\frac{2}{\alpha}} |\mathbb{P}(J_i[\mathcal{T}_h g] > M)|^{1-\frac{2}{\alpha}} \\ \le &\big|\|f\|_\infty^\alpha \|\mathcal{T}_h g\|_\infty^\alpha N_i^{2\alpha}\mathbb{E}Z_{2,i}\big|^{\frac{2}{\alpha}} \exp\left(-\frac{(1-2/\alpha)MN_i}{2\|\mathcal{T}_h g\|_\infty}\right) \\ \lesssim &\|f\|_\infty^2\|\mathcal{T}_h g\|_\infty^2 N_i^4 \exp\left(-\frac{(\alpha-2)MN_i}{2\alpha\|\mathcal{T}_h g\|_\infty}\right), \end{aligned}$$

where the second last inequality is from equation (S.17) and the last inequality is by $|A_i(f,g)-\tilde{A}_i(f,g)|^\alpha \le \|f\|_\infty^\alpha \|\mathcal{T}_h g\|_\infty^\alpha N_i^{2\alpha} Z_{2,i}$. Combing equation (S.22), (S.23) and take $f=\phi_j, g=\mathcal{P}_{\ge 2j} g$,

$$\begin{aligned}
&\mathbb{E}|A_i(\phi_j,\mathcal{P}_{\ge 2j}g)-\tilde{A}_i(\phi_j,\mathcal{P}_{\ge 2j}g)|^2\\
\le&\mathbb{E}\big[|A_i(\phi_j,\mathcal{P}_{\ge 2j}g)-\tilde{A}_i(\phi_j,\mathcal{P}_{\ge 2j}g)|^2\mathbf{1}_{(J_i[\mathcal{T}_h g]\le M)}\big]\\
&+\mathbb{E}\big[|A_i(\phi_j,\mathcal{P}_{\ge 2j}g)-\tilde{A}_i(\phi_j,\mathcal{P}_{\ge 2j}g)|^2\mathbf{1}_{(J_i[\mathcal{T}_h g]> M)}\big]\\
\lesssim&A_n^{2-\alpha}N_i^4\|\phi_j\|_\infty^2\|\mathcal{T}_h\mathcal{P}_{\ge 2j}g\|_\infty M\\
&+\|\phi_j\|_\infty^2\|\mathcal{T}_h\mathcal{P}_{\ge 2j}g\|_\infty^2 N_i^4\exp\left(-\frac{(\alpha-2)MN_i}{2\alpha\|\mathcal{T}_h\mathcal{P}_{\ge 2j}g\|_\infty}\right)\\
\lesssim&A_n^{2-\alpha}N_i^4h^{-1}M+h^{-2}N_i^4\exp\left(-\frac{(\alpha-2)MN_i}{2\alpha M_0}\right).
\end{aligned}\tag{S.24}$$

Thus,

$$\begin{aligned}
M_2^{1/2}-(M_2^0)^{1/2}\le&\sup_{g\in\chi_1(\rho)}[\mathbb{E}|A_i(\phi_j,\mathcal{P}_{\ge 2j}g)-\tilde{A}_i(\phi_j,\mathcal{P}_{\ge 2j}g)|^2]^{1/2}\\
\lesssim&A_n^{1-\alpha/2}\bar{N}_2^2|M/h|^{1/2}+h^{-1}\bar{N}_2^2\exp\left(-\frac{(\alpha-2)M\bar{N}_2}{4\alpha M_0}\right).
\end{aligned}\tag{S.25}$$

Combine equation (S.13), (S.14), (S.20) and (S.25), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{g\in\chi_1(\rho)}|\langle\hat{C}\phi_j,\mathcal{P}_{\ge 2j}g\rangle|\right]\le \mathrm{Const}\left[\sqrt{\frac{\rho\ln n}{n}\frac{M_2^0}{\bar{N}_2^4}}+A_n\frac{M}{n}\rho\ln n+h^2j^{c+1-a}\right]+\frac{\mathrm{Const}}{hA_n^{\alpha-1}}\\
&+\mathrm{Const}\sqrt{\frac{\rho\ln n}{n}}\left[A_n^{1-\alpha/2}|M/h|^{1/2}+h^{-1}\exp\left(-\frac{(\alpha-2)M\bar{N}_2}{4\alpha M_0}\right)\right]\\
&\lesssim\left[\sqrt{\frac{\rho\ln n}{n}}\left|\frac{\sqrt{M_2^0}}{\bar{N}_2^2}+\frac{1}{h}\exp\left(-\frac{(\alpha-2)M\bar{N}_2}{4\alpha M_0}\right)\right|+A_n\frac{M}{n}\rho\ln n+h^2j^{c+1-a}\right]+\frac{1}{hA_n^{\alpha-1}}.
\end{aligned}$$

Take $M=4M_1+4\frac{M_0}{\bar{N}_2}(\rho+1)\ln n+\frac{4\alpha M_0}{(\alpha-2)\bar{N}_2}\ln\frac{j^a}{h}$, $A_n=n^{1/\alpha}|Mh\rho\ln n|^{-1/\alpha}$, then,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{g\in\chi_1(\rho)}|\langle\hat{C}_0\phi_j,\mathcal{P}_{\ge 2j}g\rangle|\right]\\
\lesssim&\sqrt{\frac{\rho\ln n}{n}}\left|\frac{\sqrt{M_2^0}}{\bar{N}_2^2}+j^{-a}\right|+h^2j^{c+1-a}+\left|\frac{\rho\ln n}{n}\right|^{1-\frac{1}{\alpha}}\left|j^{1/2}+\frac{\ln n}{\bar{N}_2h}\right|^{1-\frac{1}{\alpha}}h^{-\frac{1}{\alpha}}.
\end{aligned}\tag{S.26}$$

Here we use $M_1\lesssim j^{1/2}$ and $M_0\lesssim h^{-1}$, which follow from $\|\mathcal{T}_h\mathcal{P}_{\ge 2j}g\|_1\le\|\mathcal{P}_{\ge 2j}g\|_1\le \mathrm{Const}j^{1/2}$, $\|\mathcal{T}_h\mathcal{P}_{\ge 2j}g\|_\infty\le\mathrm{Const}h^{-1}$ for all $g\in\chi_1(\rho)$. The following lemma gives the bound of M_0,

LEMMA S4. *Under Assumptions 1 to 6, $h^4j^{2a+2c}\lesssim 1$ and $hj^a\log n\lesssim 1$,*

$$M_2^0\lesssim\bar{N}_2^4j^{2-2a}+\bar{N}_2^3(j^{-a}h^{-1}+j^{2-a})+\bar{N}_2^2h^{-1}.\tag{S.27}$$

Combine equation (S.10), (S.11), (S.12), (S.26) and (S.27), on the high probability set Ω_u,

$$\mathbb{E}(\|\hat{\phi}_j - \phi_j\|_\infty) \lesssim \frac{j}{\sqrt{n}}(\sqrt{\ln n} + \ln j)\left\{1 + \frac{j^a}{\bar{N}_2} + \sqrt{\frac{j^{a-1}}{\bar{N}_2 h}}\left(1 + \sqrt{\frac{j^a}{\bar{N}_2}}\right)\right\}$$
$$+ j^a \left|\frac{\ln n}{n}\right|^{1-\frac{1}{\alpha}} \left|j^{1/2} + \frac{\ln n}{\bar{N}_2 h}\right|^{1-\frac{1}{\alpha}} h^{-\frac{1}{\alpha}} + h^2 j^{c+1} \log j$$
$$+ (j^a h^{-3} + j^{c/2}) n^{-\rho},$$

and the proof is complete by choosing large enough ρ. □

S3. Proofs of lemmas and ancillary results in main text

PROOF OF LEMMA 1. By definition of $\tilde{C}_0(s,t)$, we need to bound the bias and variance terms of

$$\iint \left\{R_{00} - C(s,t)S_{00} - h\frac{\partial C}{\partial s}(s,t)S_{10} - h\frac{\partial C}{\partial t}(s,t)S_{01}\right\} \frac{\phi_j(s)\phi_k(t)}{f(s)f(t)} \mathrm{d}s\mathrm{d}t. \tag{S.28}$$

For the random design case, by analogous calculation as proof of Theorem 3.2 in Zhang and Wang (2016), one has

$$\mathbb{E}\left[R_{00} - C(s,t)S_{00} - h\frac{\partial C}{\partial s}(s,t)S_{10} - h\frac{\partial C}{\partial t}(s,t)S_{01}\right]$$
$$= \frac{h^2}{2}\mathrm{K}_2 \frac{\partial C(s,t)}{\partial s^2}(s,t)f(s)f(t) + \frac{h^2}{2}\mathrm{K}_2 \frac{\partial C(s,t)}{\partial t^2}(s,t)f(s)f(t) + O(h^3),$$

where $\mathrm{K}_2 = \int u^2 \mathrm{K}(u)\mathrm{d}u$. Thus, for the bias part of equation (S.28), for all $k \le 2j$,

(S.29)
$$\left(\mathbb{E}\iint \left\{R_{00} - C(s,t)S_{00} - h\frac{\partial C}{\partial s}(s,t)S_{10} - h\frac{\partial C}{\partial t}(s,t)S_{01}\right\} \frac{\phi_j(s)\phi_k(t)}{f(s)f(t)} \mathrm{d}s\mathrm{d}t\right)^2$$
$$= \frac{h^4}{2}\mathrm{K}_2^2 \left[\iint \sum_{r=1}^{\infty} \lambda_r \phi_r(s)\phi_r^{(2)}(t)\phi_j(s)\phi_k(t)\mathrm{d}s\mathrm{d}t\right]^2 + o(j^{-2a}h^4)$$
$$\le \frac{h^4}{2}\mathrm{K}_2^2 \lambda_j^2 \|\phi_j\|_\infty^2 + o(h^4) = O(h^4 k^{-2a} j^{2c}).$$

For the tail summation, similarly

(S.30)
$$\sum_{k \ge j}\left(\mathbb{E}\iint \left\{R_{00} - CS_{00} - h\frac{\partial C}{\partial s}(s,t)S_{10} - h\frac{\partial C}{\partial t}(s,t)S_{01}\right\} \frac{\phi_j(s)\phi_k(t)}{f(s)f(t)} \mathrm{d}s\mathrm{d}t\right)^2$$
$$\le \frac{h^4}{2}\mathrm{K}_2^2 \left\|\int \frac{\partial^2 C}{\partial t^2}(t,s)\phi_j(s)\mathrm{d}s\right\|_{\mathrm{HS}} + o(j^{-2a}h^4)$$
$$= O(h^4 j^{1+2c-2a}).$$

For the variance of equation (S.28), note that

(S.31)

$$
\begin{aligned}
&\mathbb{V}\mathrm{ar}\left(\iint \left\{R_{00} - CS_{00} - h\frac{\partial C}{\partial s}(s,t)S_{10} - h\frac{\partial C}{\partial t}(s,t)S_{01}\right\}\frac{\phi_j(s)\phi_k(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t\right)\\
\leq &\mathbb{E}\left[\left\{\iint R_{00}(s,t)\frac{\phi_j(s)\phi_k(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t\right\}^2\right] + \mathbb{E}\left[\left\{\iint C(s,t)S_{00}(s,t)\frac{\phi_j(s)\phi_k(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t\right\}^2\right]\\
&+ h^2\mathbb{E}\left[\left\{\iint \frac{\partial C}{\partial s}(s,t)S_{10}(s,t)\frac{\phi_j(s)\phi_k(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t\right\}^2\right]\\
&+ h^2\mathbb{E}\left[\left\{\iint \frac{\partial C}{\partial t}(s,t)S_{01}(s,t)\frac{\phi_j(s)\phi_k(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t\right\}^2\right].
\end{aligned}
$$

We start with the first term in the right hand side of equation (S.31). Then

$$
\begin{aligned}
&\mathbb{E}\left[\left\{\iint R_{00}(s,t)\frac{\phi_j(s)\phi_k(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t\right\}^2\right] = \mathbb{E}\left[\left\{\sum_{i=1}^n v_i \sum_{l_1\neq l_2}^{N_i} \delta_{il_1l_2}\mathcal{T}_h\frac{\phi_j}{f}(t_{il_1})\mathcal{T}_h\frac{\phi_k}{f}(t_{il_2})\right\}^2\right]\\
=&\sum_{i=1}^n v_i^2\left\{4!\binom{N_i}{4}A_{i1} + 3!\binom{N_i}{3}A_{i2} + 2!\binom{N_i}{2}A_{i3}\right\}
\end{aligned}
$$

with

$$
\begin{aligned}
A_{i1} =&\mathbb{E}\left(\left\langle X_i f, \mathcal{T}_h\frac{\phi_j}{f}\right\rangle^2 \left\langle X_i f, \mathcal{T}_h\frac{\phi_k}{f}\right\rangle^2\right),\\
A_{i2} =&2\mathbb{E}\left\{\left(\left\langle X_i f\mathcal{T}_h\frac{\phi_j}{f}, X_i\mathcal{T}_h\frac{\phi_k}{f}\right\rangle + \sigma_X^2\left\langle \mathcal{T}_h\frac{\phi_j}{f}, f\mathcal{T}_h\frac{\phi_k}{f}\right\rangle\right)\left\langle X_i f, \mathcal{T}_h\frac{\phi_j}{f}\right\rangle\left\langle X_i f, \mathcal{T}_h\frac{\phi_k}{f}\right\rangle\right\}\\
&+\mathbb{E}\left\{\left\langle X_i f, \mathcal{T}_h\frac{\phi_j}{f}\right\rangle^2\left\langle (X_i^2+\sigma_X^2)f, \left(\mathcal{T}_h\frac{\phi_k}{f}\right)^2\right\rangle\right\}\\
&+\mathbb{E}\left\{\left\langle X_i f, \mathcal{T}_h\frac{\phi_k}{f}\right\rangle^2\left\langle (X_i^2+\sigma_X^2)f, \left(\mathcal{T}_h\frac{\phi_j}{f}\right)^2\right\rangle\right\}\\
:=&A_{i21} + A_{i22} + A_{i23}\\
A_{i3} =&\mathbb{E}\left\{\left\langle (X_i^2+\sigma_X^2)f, \left(\mathcal{T}_h\frac{\phi_j}{f}\right)^2\right\rangle\left\langle (X_i^2+\sigma_X^2)f, \left(\mathcal{T}_h\frac{\phi_k}{f}\right)^2\right\rangle\right\}\\
&+\mathbb{E}\left(\left\langle X_i\mathcal{T}_h\frac{\phi_j}{f}, X_i f\mathcal{T}_h\frac{\phi_k}{f}\right\rangle + \sigma_X^2\left\langle \mathcal{T}_h\frac{\phi_j}{f}, f\mathcal{T}_h\frac{\phi_k}{f}\right\rangle\right)^2\\
:=&A_{i31} + A_{i32}.
\end{aligned}
$$

By Cauchy–Schwarz and AM–GM inequality,

$$
\begin{aligned}
A_{i21} \leq&\mathbb{E}\left\{\left\langle X_i f, \mathcal{T}_h\frac{\phi_j}{f}\right\rangle^2\left\langle (X_i^2+\sigma_X^2)f, \left(\mathcal{T}_h\frac{\phi_k}{f}\right)^2\right\rangle\right\}\\
&+\mathbb{E}\left\{\left\langle X_i f, \mathcal{T}_h\frac{\phi_k}{f}\right\rangle^2\left\langle (X_i^2+\sigma_X^2)f, \left(\mathcal{T}_h\frac{\phi_j}{f}\right)^2\right\rangle\right\}
\end{aligned}
$$

$$=A_{i22}+A_{i23}.$$

Similarly $A_{i32} \le A_{i31}$, thus, $A_{i2} \le 2\{A_{i22}+A_{i23}\}$, $A_{i3} \le 2A_{i31}$. Combine all above,

$$\begin{aligned}&\mathbb{E}\left[\left\{\iint R_{00}(s,t)\frac{\phi_j(s)\phi_k(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t\right\}^2\right]\\ &\lesssim \sum_{i=1}^n v_i^2\left\{4!\binom{N_i}{4}A_{i1}+3!\binom{N_i}{3}(A_{i22}+A_{i23})+2!\binom{N_i}{2}A_{i31}\right\}.\end{aligned} \tag{S.32}$$

The following lemma in bounding A_{i1}, A_{i22}, A_{i23} and A_{i31}, its proof can be found in the supplement.

LEMMA S5. *Under assumptions 1 to 4 and $h^4 j^{2c+2a} \lesssim 1$, $hj^a \log n \lesssim 1$, there is $\mathbb{E}(\langle X_i f, \mathcal{T}_h \frac{\phi_k}{f}\rangle^4) \lesssim k^{-2a}$ for $1 \le k \le 2j$ and*

$$\mathbb{E}\left(\sum_{k>j}\left\langle X_i f, \mathcal{T}_h\frac{\phi_k}{f}\right\rangle^2\right)^2 \lesssim j^{2-2a}.$$

By Lemma S5 and Cauchy-Schwarz inequality

$$\begin{aligned}&A_{i1} \le \left(\mathbb{E}\left\langle X_i f, \mathcal{T}_h\frac{\phi_j}{f}\right\rangle^4 \mathbb{E}\left\langle X_i f, \mathcal{T}_h\frac{\phi_k}{f}\right\rangle^4\right)^{1/2} \lesssim j^{-a}k^{-a} \text{ and}\\ &\sum_{k>j} A_{i1} \le \left(\mathbb{E}\left\langle X_i f, \mathcal{T}_h\frac{\phi_j}{f}\right\rangle^4 \mathbb{E}\left(\sum_{k>j}\left\langle X_i f, \mathcal{T}_h\frac{\phi_k}{f}\right\rangle^2\right)^2\right)^{1/2} \lesssim j^{1-2a}.\end{aligned} \tag{S.33}$$

For A_{i22}, by Lemma S5 and Cauchy-Schwarz inequality,

$$A_{i22} \le \left\|\frac{\phi_k}{f}\right\|_\infty^2 \|f\|_\infty \left\{\mathbb{E}\left\langle X_i f, \mathcal{T}_h\frac{\phi_j}{f}\right\rangle^4 \mathbb{E}(\|X_i\|^2+\sigma_X^2)^2\right\}^{1/2} \lesssim j^{-a}, \tag{S.34}$$

For the summation $\sum_{k>j} A_{i22}$, by Cauchy-Schwarz inequality

$$\sum_{k=1}^\infty \left|\mathcal{T}_h\frac{\phi_k}{f}\right|^2 \le \frac{1}{h^2}\int\left|\mathrm{K}\left(\frac{x-y}{h}\right)\right|^2\frac{1}{f^2(y)}\mathrm{d}y \lesssim h^{-1} \text{ for all } x\in[0,1].$$

Thus,

$$\begin{aligned}\sum_{k=j}^\infty A_{i22} &= \mathbb{E}\left\{\left\langle X_i f, \mathcal{T}_h\frac{\phi_j}{f}\right\rangle^2 \sum_{k=j}^\infty\left\langle (X_i^2+\sigma_X^2)f, \left(\mathcal{T}_h\frac{\phi_k}{f}\right)^2\right\rangle\right\}\\ &\le \|f\|_\infty \sum_{k=1}^\infty\left\|\mathcal{T}_h\frac{\phi_k}{f}\right\|^2 \mathbb{E}\left\{\left\langle X_i f, \mathcal{T}_h\frac{\phi_j}{f}\right\rangle^2(\|X_i\|^2+\sigma_X^2)\right\}\\ &\lesssim h^{-1}\left(\mathbb{E}\left\langle X_i f, \mathcal{T}_h\frac{\phi_j}{f}\right\rangle^4\right)^{1/2} \lesssim h^{-1}j^{-a}.\end{aligned} \tag{S.35}$$

For A_{i23}, by Lemma S5,

$$
\begin{aligned}
A_{i23} = A_{i22} &\le \left\| \mathcal{T}_h \frac{\phi_j}{f} \right\|_\infty^2 \|f\|_\infty \mathbb{E}\left\{ \left\langle X_i f, \mathcal{T}_h \frac{\phi_k}{f} \right\rangle^2 (\|X_i\|^2 + \sigma_X^2) \right\} \\
&\lesssim \left(\mathbb{E}\left\langle X_i f, \mathcal{T}_h \frac{\phi_k}{f} \right\rangle^4 \right)^{1/2} \lesssim k^{-a}, \quad \forall\, 1 \le k \le 2j,
\end{aligned} \tag{S.36}
$$

and

$$
\begin{aligned}
\sum_{k>j} A_{i23} &\le \left\| \mathcal{T}_h \frac{\phi_j}{f} \right\|_\infty^2 \|f\|_\infty \mathbb{E}\left[\sum_{k>j} \left\langle X_i f, \mathcal{T}_h \frac{\phi_k}{f} \right\rangle^2 (\|X_i\|^2 + \sigma_X^2) \right] \\
&\lesssim \left\{ \mathbb{E}\left(\sum_{k>j} \left\langle X_i f, \mathcal{T}_h \frac{\phi_k}{f} \right\rangle^2 \right)^2 \mathbb{E}(\|X\|^2 + \sigma_X^2)^2 \right\}^{\frac{1}{2}} \lesssim j^{1-a}.
\end{aligned} \tag{S.37}
$$

For the last term A_{i31}, note that

$$
A_{i31} = \mathbb{E}\left\{ \left\langle (X_i^2 + \sigma_X^2) f, \left(\mathcal{T}_h \frac{\phi_j}{f} \right)^2 \right\rangle \left\langle (X_i^2 + \sigma_X^2) f, \left(\mathcal{T}_h \frac{\phi_k}{f} \right)^2 \right\rangle \right\} = O(1),
$$

and

$$
\begin{aligned}
\sum_{k=j}^{\infty} A_{i31} &\le \mathbb{E}\left\{ \left\langle (X_i^2 + \sigma_X^2) f, \left(\mathcal{T}_h \frac{\phi_j}{f} \right)^2 \right\rangle (\|X_i\|^2 \|f\|_\infty + \sigma_X^2 \|f\|^2) \frac{\|\mathrm{K}\|^2}{h} \right\} \\
&\lesssim h^{-1} \mathbb{E}(\|X_i\|^2 \|f\|_\infty + \sigma_X^2 \|f\|_\infty)^2 \lesssim h^{-1}.
\end{aligned} \tag{S.38}
$$

Combine equation (S.33) to (S.38), for all $k \le 2j$

$$
\mathbb{E}\left[\left\{ \iint R_{00}(s,t) \frac{\phi_j(s)\phi_k(t)}{f(s)f(t)} \mathrm{d}s\mathrm{d}t \right\}^2 \right] \lesssim \frac{1}{n}\left(j^{-a}k^{-a} + \frac{j^{-a} + k^{-a}}{\bar{N}_2} + \frac{1}{\bar{N}_2^2} \right)
$$

and

$$
\sum_{k>j} \mathbb{E}\left[\left\{ \iint R_{00}(s,t) \frac{\phi_j(s)\phi_k(t)}{f(s)f(t)} \mathrm{d}s\mathrm{d}t \right\}^2 \right] \lesssim \frac{1}{n}\left(j^{1-2a} + \frac{j^{-a}h^{-1} + j^{1-a}}{\bar{N}_2} + \frac{h^{-1}}{\bar{N}_2^2} \right).
$$

By similar analysis, the second term in the right hand side of equation (S.31) has the same convergence rate as the first term. Under $h^4 j^{2a+2c} \lesssim 1$ and $hj^a \log n \lesssim 1$, the last two terms in the right hand side of equation (S.31) are dominated by the first two terms. Then the proof of the random design case is complete by

$$
\begin{aligned}
&\mathbb{E}\left(\iint \left\{ R_{00} - C(s,t)S_{00} - h\frac{\partial C}{\partial s}(s,t)S_{10} - h\frac{\partial C}{\partial t}(s,t)S_{01} \right\} \frac{\phi_j(s)\phi_k(t)}{f(s)f(t)} \mathrm{d}s\mathrm{d}t \right)^2 \\
\lesssim& \frac{1}{n}\left(j^{-a}k^{-a} + \frac{j^{-a} + k^{-a}}{\bar{N}_2} + \frac{1}{\bar{N}_2^2} \right) + h^4 k^{-2a} j^{2c}
\end{aligned}
$$

for all $k \le 2j$ and

$$
\begin{aligned}
&\sum_{k>j} \mathbb{E}\left(\iint \left\{ R_{00} - C(s,t)S_{00} - h\frac{\partial C}{\partial s}(s,t)S_{10} - h\frac{\partial C}{\partial t}(s,t)S_{01} \right\} \frac{\phi_j(s)\phi_k(t)}{f(s)f(t)} \mathrm{d}s\mathrm{d}t \right)^2 \\
\lesssim& \frac{1}{n}\left(j^{1-2a} + \frac{j^{-a}h^{-1} + j^{1-a}}{\bar{N}_2} + \frac{h^{-1}}{\bar{N}_2^2} \right) + h^4 j^{1-2a+2c}.
\end{aligned}
$$

□

To prove the fixed-design case of Theorem 1, it is sufficient to verify the following lemma and proposition, which are the fixed-design versions of Lemma 1 and the proposition in the main text.

LEMMA S6. *Under assumptions 1 to 4, 7 and 8, for all* $1 \le k \le 2j$,

$$\mathbb{E}\left[\left\{\iint \tilde{C}_0(s,t)\phi_j(s)\phi_k(t)\mathrm{d}s\mathrm{d}t\right\}^2\right] \lesssim \frac{1}{n}\left(j^{-a}k^{-a} + \frac{j^{-a}+k^{-a}}{N} + \frac{1}{N^2}\right) + h^4k^{2c-2a}$$

and

$$\sum_{k=j+1}^{\infty}\mathbb{E}\left[\left\{\iint \tilde{C}_0(s,t)\phi_j(s)\phi_k(t)\mathrm{d}s\mathrm{d}t\right\}^2\right]$$
$$\lesssim\frac{1}{n}\left(j^{1-2a} + \frac{h^{-1}j^{-a}+j^{1-a}}{N} + \frac{1}{hN^2}\right) + h^4j^{1+2c-2a}.$$

PROPOSITION S1. *Under Assumption 2, 7 and 8,*

(a) $\inf_{s,t} S_{00}(s,t)I_1(s,t) + S_{10}(s,t)I_2(s,t) + S_{01}(s,t)I_3(s,t)$ *and* $\inf_{s,t} S_{00}(s,t)$ *are bounded away from zero alost surely.*

(b)

$$\left\|S_{00}(s,t) - 1_{\{0\le s\le 1, 0\le t\le 1\}}\right\|_{\mathrm{HS}}^2 = O_P\left(h^2 + \frac{1}{n}\left\{1 + \frac{1}{\bar{N}_2^2h^2}\right\}\right).$$

(c) *For* $p,q = 0,1,2$, $\|S_{pq}\|_\infty = O(1)$ *a.s.*

(d) *For* $p,q = 0,1$, $p+q=1$

$$\|S_{pq}(s,t)\|_{\mathrm{HS}}^2 = O_P\left(h + \frac{\log n}{n}\left\{1 + \frac{1}{\bar{N}_2^2h^2}\right\}\right).$$

(e) *In addition, if Assumption 5 holds with* $\alpha > 3$, *for* $p,q = 0,1$

$$\sup_{s,t}\left|R_{pq}(s,t) - C(s,t)S_{pq}(s,t) - h\frac{\partial C(s,t)}{\partial s}S_{p+1,q}(s,t) - h\frac{\partial C(s,t)}{\partial t}S_{p,q+1}(s,t)\right|$$
$$=O\left(h^2 + \sqrt{\frac{\log n}{n}\left\{1 + \frac{1}{\bar{N}_2^2h^2}\right\}}\right) \text{ a.s.}$$

For the proof of Proposition 1 and S1, the statements (a) and (c) can be verified by Lemma S.4.3 and S.4.2 in Shao, Lin and Yao (2022), while statements (b) and (d) can be justified with arguments analogous to those in Zhang and Wang (2016). We thus omit the proofs.

PROOF OF LEMMA S6. For the even grid design case, by Taylor expansion and similar arguments,

$$\mathbb{E}\iint\left\{R_{00} - CS_{00} - h\frac{\partial C}{\partial s}S_{10} - h\frac{\partial C}{\partial t}S_{01}\right\}\phi_j(s)\phi_k(t)\mathrm{d}s\mathrm{d}t$$
$$=\sum_{i=1}^{n}v_i\sum_{l_1\ne l_2}\frac{1}{h}\int \mathrm{K}\left(\frac{t_{il_1}-s}{h}\right)\phi_j(s)\mathrm{d}s\frac{1}{h}\int K\left(\frac{t_{il_2}-t}{h}\right)\phi_k(t)\mathrm{d}tC(t_{il1},t_{il2})$$

$$
\begin{aligned}
&-\sum_{i=1}^{n} v_i \sum_{l_1\neq l_2} \frac{1}{h^2}\iint \mathrm{K}\left(\frac{t_{il_1}-s}{h}\right) K\left(\frac{t_{il_2}-t}{h}\right)\phi_k(t)\phi_j(s)C(s,t)\mathrm{d}t\mathrm{d}s\\
&-h\sum_{i=1}^{n} v_i \sum_{l_1\neq l_2} \frac{1}{h^2}\iint \mathrm{K}\left(\frac{t_{il_1}-s}{h}\right) K\left(\frac{t_{il_2}-t}{h}\right)\phi_k(t)\phi_j(s)\frac{\partial C}{\partial s}(s,t)\mathrm{d}t\mathrm{d}s\\
&-h\sum_{i=1}^{n} v_i \sum_{l_1\neq l_2} \frac{1}{h^2}\iint \mathrm{K}\left(\frac{t_{il_1}-s}{h}\right) K\left(\frac{t_{il_2}-t}{h}\right)\phi_k(t)\phi_j(s)\frac{\partial C}{\partial t}(s,t)\mathrm{d}t\mathrm{d}s\\
=&h^2\frac{1}{N(N-1)}\sum_{l_1\neq l_2}\phi_j(t_{il_1})\phi_k(t_{il_2})\frac{\partial^2 C(s,t)}{\partial t^2}(t_{il_1},t_{il_2})+o(h^2)\\
=&h^2\iint \phi_j(u)\phi_k(v)\frac{\partial^2 C(u,v)}{\partial v^2}(u,v)\mathrm{d}u\mathrm{d}v+O(N^{-2})+o(h^2)\\
=&O(h^2 j^{c-a}),
\end{aligned}
$$

where the last two equalities are due to the Riemann sum approximation and assumptions $Nh \gtrsim 1$ and $hj^a \log n \lesssim 1$. Similarly,

$$
\begin{aligned}
&\sum_{k>j}\left(\mathbb{E}\iint\left\{R_{00}-CS_{00}-h\frac{\partial C}{\partial s}S_{10}-h\frac{\partial C}{\partial t}S_{01}\right\}\phi_j(s)\phi_k(t)\mathrm{d}s\mathrm{d}t\right)^2\\
\lesssim&\sum_{k>j}h^2\iint \phi_j(u)\phi_k(v)\frac{\partial^2 C(u,v)}{\partial v^2}(u,v)\mathrm{d}u\mathrm{d}v+O(N^{-2})+o(h^4)\lesssim h^4 j^{1+2c-2a}.
\end{aligned}
$$

For the variance term, by similar analysis, it is enough to bound

$$
\begin{aligned}
&\mathbb{E}\left[\left\{\iint R_{00}(s,t)\phi_j(s)\phi_k(t)\mathrm{d}s\mathrm{d}t\right\}^2\right]\\
=&\sum_{i=1}^{n} v_i^2\mathbb{E}\Bigg(\sum_{l_1\neq l_2\neq l_3\neq l_4}\mathcal{T}_h\phi_j(t_{il_1})X_i(t_{il_1})\mathcal{T}_h\phi_k(t_{il_2})X_i(t_{il_2})\mathcal{T}_h\phi_j(t_{il_3})X_i(t_{il_3})\mathcal{T}_h\phi_k(t_{il_4})X_i(t_{il_4})\\
&+\sum_{l_1\neq l_2\neq l_3}[\mathcal{T}_h\phi_j(t_{il_1})X_i(t_{il_1})\mathcal{T}_h\phi_j(t_{il_2})X_i(t_{il_2})\left\{X_i^2(t_{il_3})+\sigma_X^2\right\}\mathcal{T}_h\phi_j(t_{il_3})^2\\
&+\sum_{l_1\neq l_2\neq l_3}\mathcal{T}_h\phi_k(t_{il_1})X_i(t_{il_1})\mathcal{T}_h\phi_k(t_{il_2})X_i(t_{il_2})\left\{X_i^2(t_{il_3})+\sigma_X^2\right\}\mathcal{T}_h\phi_k(t_{il_3})^2\\
&+\sum_{l_1\neq l_2\neq l_3}2\mathcal{T}_h\phi_j(t_{il_1})X_i(t_{il_1})\mathcal{T}_h\phi_k(t_{il_2})X_i(t_{il_2})\mathcal{T}_h\phi_j(t_{il_3})\mathcal{T}_h\phi_k(t_{il_3})\{X_i^2(t_{il_3})+\sigma_X^2\}\\
&+\sum_{l_1\neq l_2}\left\{X_i^2(t_{il_1})+\sigma_X^2\right\}\mathcal{T}_h\phi_j(t_{il_1})^2\left\{X_i^2(t_{il_2})+\sigma_X^2\right\}\mathcal{T}_h\phi_k(t_{il_2})^2\\
&+\sum_{l_1\neq l_2}\left\{X_i^2(t_{il_1})+\sigma_X^2\right\}\mathcal{T}_h\phi_j(t_{il_1})\mathcal{T}_h\phi_k(t_{il_1})\left\{X_i^2(t_{il_2})+\sigma_X^2\right\}\mathcal{T}_h\phi_j(t_{il_2})\mathcal{T}_h\phi_k(t_{il_2})\Bigg).
\end{aligned}
$$

By the approximation of Riemann sums, there is

$$
\begin{aligned}
&\mathbb{E}\left[\left\{\iint R_{00}(s,t)\phi_j(s)\phi_k(t)\mathrm{d}s\mathrm{d}t\right\}^2\right] \\
\lesssim &\sum_{i=1}^n v_i^2 N^4 \mathbb{E}\left(\langle X_i, \mathcal{T}_h\phi_j\rangle + \frac{1}{N}\right)^2 \mathbb{E}\left(\langle X_i, \mathcal{T}_h\phi_k\rangle + \frac{1}{N}\right)^2 \\
&+ 2\sum_{i=1}^n v_i^2 N^3 \mathbb{E}\left(\langle X_i, \mathcal{T}_h\phi_j\rangle + \frac{1}{N}\right)\left(\langle X_i, \mathcal{T}_h\phi_k\rangle + \frac{1}{N}\right) \\
&\qquad\qquad \times \left(\langle X_i\mathcal{T}_h\phi_j, X_i\mathcal{T}_h\phi_k\rangle + \sigma_X^2\langle \mathcal{T}_h\phi_j, \mathcal{T}_h\phi_k\rangle + \frac{1}{N}\right) \\
&+ \sum_{i=1}^n v_i^2 N^3 \mathbb{E}\left(\langle X_i, \mathcal{T}_h\phi_j\rangle + \frac{1}{N}\right)^2\left(\|X_i\mathcal{T}_h\phi_k\|^2 + \sigma_X^2\|\mathcal{T}_h\phi_k\|^2 + \frac{1}{N}\right) \\
&+ \sum_{i=1}^n v_i^2 N^3 \mathbb{E}\left(\langle X_i, \mathcal{T}_h\phi_k\rangle + \frac{1}{N}\right)^2\left(\|X_i\mathcal{T}_h\phi_j\|^2 + \sigma_X^2\|\mathcal{T}_h\phi_j\|^2 + \frac{1}{N}\right) \\
&+ \sum_{i=1}^n v_i^2 N^2 \mathbb{E}\left(\|X_i\mathcal{T}_h\phi_j\|^2 + \sigma_X^2\|\mathcal{T}_h\phi_j\|^2 + \frac{1}{N}\right)\left(\|X_i\mathcal{T}_h\phi_k\|^2 + \sigma_X^2\|\mathcal{T}_h\phi_k\|^2 + \frac{1}{N}\right) \\
&+ \sum_{i=1}^n v_i^2 N^2 \mathbb{E}\left(\langle X_i\mathcal{T}_h\phi_j, X_i\mathcal{T}_h\phi_k\rangle + \sigma_X^2\langle \mathcal{T}_h\phi_j, \mathcal{T}_h\phi_k\rangle + \frac{1}{N}\right)^2.
\end{aligned}
$$

Under the assumption $Nh \gtrsim 1$ and $hj^a \log n \lesssim 1$, it is not difficult to verify $\mathbb{E}[\{\iint R_{00}(s,t)\phi_j(s)\phi_k(t)\mathrm{d}s\mathrm{d}t\}^2]$ is dominated by

$$
\begin{aligned}
&\frac{1}{n}\left\{\mathbb{E}(\langle X_i, \mathcal{T}_h\phi_j\rangle^2\langle X_i, \mathcal{T}_h\phi_k\rangle^2) + \frac{\mathbb{E}\{\langle X_i, \mathcal{T}_h\phi_j\rangle^2(\|X_i\mathcal{T}_h\phi_k\|^2 + \sigma_X^2\|\mathcal{T}_h\phi_k\|^2)\}}{N}\right. \\
&+ \frac{\mathbb{E}\{\langle X_i, \mathcal{T}_h\phi_k\rangle^2(\|X_i\mathcal{T}_h\phi_j\|^2 + \sigma_X^2\|\mathcal{T}_h\phi_j\|^2)\}}{N} \\
&\left.+ \frac{\mathbb{E}\{(\|X_i\mathcal{T}_h\phi_j\|^2 + \sigma_X^2\|\mathcal{T}_h\phi_j\|^2)(\|X_i\mathcal{T}_h\phi_k\|^2 + \sigma_X^2\|\mathcal{T}_h\phi_k\|^2)\}}{N^2}\right\}.
\end{aligned}
$$

By similar arguments as Lemma S5, we have the following lemma.

PROPOSITION S2. *Under Assumptions 1 to 4, 6, 7 and* $h^4 j^{2c+2a} \lesssim 1$, $hj^a \log n \lesssim 1$, $\mathbb{E}(\langle X_i, \mathcal{T}_h\phi_k\rangle^4) \lesssim k^{-2a}$ *for* $1 \le k \le 2j$ *and*

$$
\mathbb{E}\left(\sum_{k>j}\langle X_i, \mathcal{T}_h\phi_k\rangle^2\right)^2 \lesssim j^{2-2a}.
$$

By similar analysis as in Lemma 2, under the assumptions $Nh \gtrsim 1$ and $hj^a \log n \lesssim 1$, we finally obtain

$$
\begin{aligned}
&\mathbb{E}\left(\iint\left\{R_{00} - C(s,t)S_{00} - h\frac{\partial C}{\partial s}(s,t)S_{10} - h\frac{\partial C}{\partial t}(s,t)S_{01}\right\}\phi_j(s)\phi_k(t)\mathrm{d}s\mathrm{d}t\right)^2 \\
\lesssim &\frac{j^{-a}k^{-a}}{n} + h^4 k^{-2a} j^{2c}
\end{aligned}
$$

for all $k \le 2j$ and

$$\sum_{k>j}\mathbb{E}\left(\iint\left\{R_{00}-C(s,t)S_{00}-h\frac{\partial C}{\partial s}(s,t)S_{10}-h\frac{\partial C}{\partial t}(s,t)S_{01}\right\}\phi_j(s)\phi_k(t)\mathrm{d}s\mathrm{d}t\right)^2$$
$$\lesssim\frac{j^{1-2a}}{n}+h^4j^{1-2a+2c},$$

which complete the proof. □

PROOF OF COROLLARY 3. When the mean function $\mu(t)$ is unknown, the raw covariances in the local linear smoother are $\tilde{\delta}_{il_1l_2}=\{X_{il_1}-\hat{\mu}(t_{il_1})\}\{X_{il_2}-\hat{\mu}(t_{il_2})\}$, where $\hat{\mu}(t)$ is estimated by

$$\hat{\mu}(t)=\arg\min_{\beta_0}\sum_{i=1}^n v_i\sum_{j=1}^{N_i}\{X_{ij}-\beta_0-\beta_1(t_{ij}-t)\}\frac{1}{h_\mu}\mathrm{K}\left(\frac{t_{ij}-t}{h_\mu}\right).$$

Denote

$$\tilde{R}_{p,q}(s,t)=\sum_{i=1}^n v_i\sum_{l_1\ne l_2}^{N_i}\frac{1}{h^2}\mathrm{K}\left(\frac{t_{il_1}-s}{h}\right)\mathrm{K}\left(\frac{t_{il_2}-t}{h}\right)(t_{il_1}-s)^p(t_{il_2}-t)^q\tilde{\delta}_{il_1l_2}.$$

By equation (19) in the main text, it is enough to bound

$$\left[\iint\left\{\tilde{R}_{00}(s,t)-R_{00}(s,t)\right\}\frac{\phi_j(s)\phi_k(t)}{f(s)f(t)}\mathrm{d}s\mathrm{d}t\right]^2$$
$$\le 3\left[\sum_{i=1}^n v_i\sum_{l_1\ne l_2}^{N_i}\mathcal{T}_h\frac{\phi_j}{f}(t_{il_1})\mathcal{T}_h\frac{\phi_k}{f}(t_{il_2})\{\hat{\mu}(t_{il_1})-\mu(t_{il_1})\}\{X_{il_2}-\mu(t_{il_2})\}\right]^2$$
$$+3\left[\sum_{i=1}^n v_i\sum_{l_1\ne l_2}^{N_i}\mathcal{T}_h\frac{\phi_j}{f}(t_{il_1})\mathcal{T}_h\frac{\phi_k}{f}(t_{il_2})\{X_{il_1}-\mu(t_{il_1})\}\{\hat{\mu}(t_{il_2})-\mu(t_{il_2})\}\right]^2 \tag{S.39}$$
$$+3\left[\sum_{i=1}^n v_i\sum_{l_1\ne l_2}^{N_i}\mathcal{T}_h\frac{\phi_j}{f}(t_{il_1})\mathcal{T}_h\frac{\phi_k}{f}(t_{il_2})\{\hat{\mu}(t_{il_1})-\mu(t_{il_1})\}\{\hat{\mu}(t_{il_2})-\mu(t_{il_2})\}\right]^2.$$

For the first term in the right hand side of equation (S.39),

$$\left[\sum_{i=1}^n v_i\sum_{l_1\ne l_2}^{N_i}\mathcal{T}_h\frac{\phi_j}{f}(t_{il_1})\mathcal{T}_h\frac{\phi_k}{f}(t_{il_2})\{\hat{\mu}(t_{il_1})-\mu(t_{il_1})\}\{X_{il_2}-\mu(t_{il_2})\}\right]^2$$
$$=\left[\sum_{i=1}^n v_i\sum_{l_1=1}^{N_i}\mathcal{T}_h\frac{\phi_j}{f}(t_{il_1})\{\hat{\mu}(t_{il_1})-\mu(t_{il_1})\}\sum_{l_2\ne l_1}^{N_i}\mathcal{T}_h\frac{\phi_k}{f}(t_{il_2})\{X_{il_2}-\mu(t_{il_2})\}\right]^2$$
$$\le\mathbb{E}\|\hat{\mu}-\mu\|_\infty^2\|\mathrm{K}\|_\infty^2\|\phi_j/f\|_\infty^2\left[\frac{1}{n}\sum_{i=1}^n\frac{1}{N_i-1}\sum_{l_2\ne l_1}^{N_i}\{X_{il_2}-\mu(t_{il_2})\}\mathcal{T}_h\frac{\phi_k}{f}(t_{il_2})\right]^2$$

Note that

$$\mathbb{E}\left[\sum_{l_2\neq l_1}^{N_i}\{X_{il_2}-\mu(t_{il_2})\}\mathcal{T}_h\frac{\phi_k}{f}(t_{il_2})\right]^2=\sum_{l_2\neq l_1}\mathbb{E}\{X_{il_2}-\mu(t_{il_2})\}^2\left\{\mathcal{T}_h\frac{\phi_k}{f}(t_{il_2})\right\}^2$$
$$=O_P((N_i-1)k^{-a}).$$

By Theorem 5.1 in Zhang and Wang (2016),

$$\|\hat{\mu}-\mu\|_\infty=O\left(\sqrt{\frac{\log n}{n}\left(1+\frac{1}{\bar{N}_1 h_\mu}\right)}+h_\mu^2\right)\text{ a.s. },$$

where $\bar{N}_1=(n^{-1}\sum_{i=1}^n N_i^{-1})^{-1}$.

Thus,

$$\left[\sum_{i=1}^n v_i\sum_{l_1\neq l_2}^{N_i}\mathcal{T}_h\frac{\phi_j}{f}(t_{il_1})\mathcal{T}_h\frac{\phi_k}{f}(t_{il_2})\{\hat{\mu}(t_{il_1})-\mu(t_{il_1})\}\{X_{il_2}-\mu(t_{il_2})\}\right]^2$$
$$=O_P\left(\frac{k^{-a}}{\bar{N}_1}\left\{\frac{\log n}{n}\left(1+\frac{1}{\bar{N}_1 h_\mu}\right)+h_\mu^4\right\}\right).$$

By similar analysis, the second term in the right hand side of equation (S.39) has the same bound as the first term. For the last term,

$$\left[\sum_{i=1}^n v_i\sum_{l_1\neq l_2}^{N_i}\mathcal{T}_h\frac{\phi_j}{f}(t_{il_1})\mathcal{T}_h\frac{\phi_k}{f}(t_{il_2})\{\hat{\mu}(t_{il_1})-\mu(t_{il_1})\}\{\mu(t_{il_1})-\mu(t_{il_2})\}\right]^2$$
$$\leq\|K\|_\infty^4\|\phi_j/f\|_\infty^2\|\phi_k/f\|_\infty^2\|\hat{\mu}-\mu\|_\infty^4,$$

which is dominated by the first two terms. Thus, under the assumption $\bar{N}_1\gtrsim j^a$ and $\bar{N}_1 h_\mu\gtrsim 1$, $[\iint\{\tilde{R}_{00}-R_{00}\}\phi_j(s)\phi_k(t)/(f(s)f(t))\mathrm{d}s\mathrm{d}t]^2$ is dominated by

$$\frac{1}{n}\left(j^{-a}k^{-a}+\frac{j^{-a}+k^{-a}}{\bar{N}_2}+\frac{1}{\bar{N}_2^2}\right)+h^4k^{2c-2a}.$$

Similarly, under the assumption $\bar{N}_1\gtrsim j^a$ and $hj^{2a}\log n\lesssim 1$,

$$\sum_{k>j}\left[\iint\{\tilde{R}_{00}-R_{00}\}\phi_j(s)\phi_k(t)/(f(s)f(t))\mathrm{d}s\mathrm{d}t\right]^2$$

is dominated by

$$O_P\left(\frac{1}{n}\left(j^{1-2a}+\frac{j^{-a}h^{-1}+j^{1-a}}{\bar{N}_2}+\frac{h^{-1}}{\bar{N}_2^2}\right)+h^4j^{1-2a+2c}\right).$$

□

S4. Proofs of lemmas in supplement

PROOF OF LEMMA S1 . As $(\hat{\lambda}_j-\lambda_k)\langle\hat{\phi}_j,\phi_k\rangle=\int\hat{C}_0\hat{\phi}_j\phi_k-\int C\hat{\phi}_j\phi_k=\int(\hat{C}_0-C)\phi_j\phi_k+\int(\hat{C}_0-C)(\hat{\phi}_j-\phi_j)\phi_k$, by expansion in Hsing and Eubank (2015), we have

(S.40)

$$\hat{\phi}_j-\phi_j=\sum_{k\neq j}\frac{\int(\hat{C}_0-C)\phi_j\phi_k}{\hat{\lambda}_j-\lambda_k}\phi_k+\sum_{k\neq j}\frac{\int(\hat{C}_0-C)(\hat{\phi}_j-\phi_j)\phi_k}{\hat{\lambda}_j-\lambda_k}\phi_k+\left\{\int(\hat{\phi}_j-\phi_j)\phi_j\right\}\phi_j$$
$$:=I_1+I_2+I_3.$$

The last term in equation (S.40) is bounded by

$$\|I_3\|_\infty \le \|\hat{\phi}_j - \phi_j\|_2 \|\phi_j\|_2 \|\phi_j\|_\infty \lesssim \|\hat{\phi}_j - \phi_j\|_2. \tag{S.41}$$

For I_2, note that

$$I_2 = \frac{(\hat{C}_0 - C)(\hat{\phi}_j - \phi_j)}{\hat{\lambda}_j} + \sum_{k\neq j} \frac{\lambda_k \int (\hat{C}_0 - C)(\hat{\phi}_j - \phi_j)\phi_k}{\hat{\lambda}_j(\hat{\lambda}_j - \lambda_k)} \phi_k := I_{2,1} + I_{2,2}, \tag{S.42}$$

where

$$\|I_{2,1}\|_\infty = \|(\hat{C}_0 - C)(\hat{\phi}_j - \phi_j)\|_\infty / \hat{\lambda}_j \lesssim \|(\hat{C}_0 - C)\|_\infty \|\hat{\phi}_j - \phi_j\|_2 j^a,$$

and

$$\|I_{2,2}\|_\infty \le \sum_{k\neq j} \frac{\lambda_k |\int (\hat{C}_0 - C)(\hat{\phi}_j - \phi_j)\phi_k|}{\hat{\lambda}_j |\hat{\lambda}_j - \lambda_k|} \|\phi_k\|_\infty \lesssim \sum_{k\neq j} \frac{\lambda_k |\int (\hat{C}_0 - C)(\hat{\phi}_j - \phi_j)\phi_k|}{\hat{\lambda}_j |\hat{\lambda}_j - \lambda_k|}$$

$$\lesssim \left(\sum_{k\neq j} \left| \int (\hat{C}_0 - C)(\hat{\phi}_j - \phi_j)\phi_k \right|^2 \right)^{\frac{1}{2}} \left(\sum_{k\neq j} \frac{\lambda_k^2}{\hat{\lambda}_j^2 |\hat{\lambda}_j - \lambda_k|} \right)^{\frac{1}{2}} \lesssim \|(\hat{C}_0 - C)(\hat{\phi}_j - \phi_j)\|_2 j^{a+1}$$

$$\lesssim \|(\hat{C}_0 - C)\|_{\mathrm{HS}} \|\hat{\phi}_j - \phi_j\|_2 j^{a+1}.$$

Next, we focus on I_1, which is the dominating term, by similar arguments,

$$I_1 = \sum_{k\neq j, k<2j} \frac{\int (\hat{C}_0 - C)\phi_j \phi_k}{\hat{\lambda}_j - \lambda_k} \phi_k + \sum_{k=2j}^{\infty} \frac{\lambda_k \int (\hat{C}_0 - C)\phi_j\phi_k}{\hat{\lambda}_j(\hat{\lambda}_j - \lambda_k)} \phi_k + \sum_{k=2j}^{\infty} \frac{\int (\hat{C}_0 - C)\phi_j\phi_k}{\hat{\lambda}_j} \phi_k$$

$$:= I_{1,1} + I_{1,2} + I_{1,3}.$$

For $I_{1,1}$ and $I_{1,2}$, by Theorem 1, there are

$$\|I_{1,1}\|_\infty \le \sum_{k\neq j, k<2j} \frac{|\int (\hat{C}_0 - C)\phi_j\phi_k|}{|\hat{\lambda}_j - \lambda_k|} \|\phi_k\|_\infty \lesssim \sum_{k\neq j, k<2j} \frac{|\int (\hat{C}_0 - C)\phi_j\phi_k|}{|\hat{\lambda}_j - \lambda_k|}$$

$$\lesssim \sum_{j/2<k<2j} k^{a+1} \frac{1}{|k-j|} \left\{ \frac{1}{\sqrt{n}} \left(j^{-\frac{a}{2}} k^{-\frac{a}{2}} + \frac{j^{-\frac{a}{2}} + k^{-\frac{a}{2}}}{\sqrt{\bar{N}_2}} + \frac{1}{\bar{N}_2} \right) + h^2 k^{c-a} \right\}$$

$$= O_p \left(\frac{j \log j}{\sqrt{n}} + \frac{1}{\sqrt{n \bar{N}_2}} j^{\frac{a}{2}+1} \log j + \frac{1}{\sqrt{n} \bar{N}_2} j^{a+1} \log j + h^2 j^{c+1} \log j \right)$$

and

$$\|I_{1,2}\|_\infty \le \sum_{k=2j}^{\infty} \frac{\lambda_k |\int (\hat{C}_0 - C)\phi_j\phi_k|}{\hat{\lambda}_j |\hat{\lambda}_j - \lambda_k|} \|\phi_k\|_\infty$$

$$\lesssim \left(\sum_{k=2j}^{\infty} \left| \int (\hat{C}_0 - C)\phi_j\phi_k \right|^2 \right)^{\frac{1}{2}} \left(\sum_{k=2j}^{\infty} \frac{\lambda_k^2}{\hat{\lambda}_j^2 |\hat{\lambda}_j - \lambda_k|} \right)^{\frac{1}{2}}$$

$$\lesssim \left(\sum_{k=2j}^{\infty} \left| \int (\hat{C}_0 - C)\phi_j\phi_k \right|^2 \right)^{\frac{1}{2}} j^{a+1/2}$$

$$= O_p \left(\frac{1}{\sqrt{n}} \left(j + \frac{h^{-\frac{1}{2}} j^{\frac{a+1}{2}} + j^{1+\frac{a}{2}}}{\sqrt{\bar{N}_2}} + \frac{j^{a+\frac{1}{2}}}{\sqrt{h} \bar{N}_2} \right) + h^2 j^{1+c} \right).$$

In summary, there are

$$\hat{\phi}_j - \phi_j = I_1 + I_2 + I_3 = I_{1,1} + I_{1,2} + I_{1,3} + I_{2,1} + I_{2,2} + I_3 = \phi_{j,0} + \hat{\lambda}_j^{-1}\phi_{j,1},$$

$$\phi_{j,0} := I_{1,1} + I_{1,2} + I_{2,1} + I_{2,2} + I_3, \quad \phi_{j,1} := \hat{\lambda}_j I_{1,3} = \sum_{k=2j}^{\infty} \left\{ \int (\hat{C}_0 - C)\phi_j\phi_k \right\} \phi_k.$$

On the high probability set Ω_u, $\|(\hat{C}_0 - C)\|_\infty j^a + \|(\hat{C}_0 - C)\|_{\text{HS}} j^{a+1} = O(1)$ then

$$\begin{aligned}
&\|\phi_{j,0}\|_\infty \le \|I_{1,1}\|_\infty + \|I_{1,2}\|_\infty + \|I_{2,1}\|_\infty + \|I_{2,2}\|_\infty + \|I_3\|_\infty \\
\lesssim &\|I_{1,1}\|_\infty + \|I_{1,2}\|_\infty + (1 + \|\Delta(\hat{C}_0 - C)\|_\infty j^a + \|(\hat{C}_0 - C)\|_{\text{HS}} j^{a+1})\|\hat{\phi}_j - \phi_j\|_2 \\
\lesssim &\|I_{1,1}\|_\infty + \|I_{1,2}\|_\infty + \|\hat{\phi}_j - \phi_j\|_2 \\
= &O_p\Bigg(\frac{j\log j}{\sqrt{n}} + \frac{1}{\sqrt{n\bar{N}_2}}\left(j^{\frac{a}{2}+1}\log j + h^{-\frac{1}{2}} j^{\frac{a+1}{2}} \right) \\
&\qquad + \frac{j^{a+\frac{1}{2}}}{\sqrt{nh}\bar{N}_2} + \frac{1}{\sqrt{n}\bar{N}_2} j^{a+1}\log j + h^2 j^{c+1}\log j \Bigg),
\end{aligned}$$

which completes the proof. □

PROOF OF LEMMA S2. For the first statement in Lemma 4, as $\mathcal{P}_{<m} g = \sum_{k=1}^{m-1}\langle g, \phi_k\rangle\phi_k$,

$$\begin{aligned}
&\|\mathcal{P}_{<m} g\|_2^2 = \sum_{k=1}^{m-1} |\langle g, \phi_k\rangle|^2 \le \sum_{k=1}^{m-1} \|g\|_1^2 \|\phi_k\|_\infty^2 \lesssim \sum_{k=1}^{m-1} 1 \lesssim m, \\
&\|\mathcal{P}_{<m} g\|_\infty \le \sum_{k=1}^{m-1} |\langle g, \phi_k\rangle| \|\phi_k\|_\infty \le \sum_{k=1}^{m-1} \|g\|_1 \|\phi_k\|_\infty^2 \lesssim \sum_{k=1}^{m-1} 1 \lesssim m, \\
&\|\mathcal{P}_{\ge 2m} g\|_1 = \|g - \mathcal{P}_{<2m} g\|_1 \le \|g\|_1 + \|\mathcal{P}_{<2m} g\|_2 \lesssim 1 + (2m)^{1/2} \lesssim m^{1/2}.
\end{aligned}$$

For the second statement, note that

$$\left| \int C(\mathcal{T}_h\phi_j - \phi_j)\phi_k \right| = \lambda_k |\langle (\mathcal{T}_h\phi_j - \phi_j), \phi_k\rangle| \le \lambda_k \|\mathcal{T}_h\phi_j - \phi_j\|_2 \|\phi_k\|_2 \lesssim k^{-a} h^2 j^c,$$

and for all $f \in \mathcal{L}^2$,

$$\begin{aligned}
&\|(\mathcal{T}_h C - C) f\|_\infty = \left\| \sum_{k=1}^{\infty} \lambda_k (\mathcal{T}_h\phi_k - \phi_k)\langle f, \phi_k\rangle \right\|_\infty \le \sum_{k=1}^{\infty} \lambda_k \|\mathcal{T}_h\phi_k - \phi_k\|_\infty |\langle f, \phi_k\rangle| \\
&\lesssim \sum_{k=1}^{\infty} k^{-a} \min(1, h^2 k^c) |\langle f, \phi_k\rangle| \\
&\lesssim \left[\sum_{k=1}^{\infty} k^{-2a} \min(1, h^4 k^{2c}) \right]^{1/2} \left[\sum_{k=1}^{\infty} |\langle f, \phi_k\rangle|^2 \right]^{1/2} \lesssim j^{1/2-a} \|f\|_2.
\end{aligned}$$

Recall $\|g\|_1 = 1$ for $g \in \chi_1(\rho)$,

$$
\begin{aligned}
&\langle \mathbb{E}\hat{C}_0\phi_j, \mathcal{P}_{\geq 2j} g\rangle = \langle \mathcal{T}_h C \mathcal{T}_h \phi_j, \mathcal{P}_{\geq 2j} g\rangle = \langle (\mathcal{T}_h C - C)(\mathcal{T}_h\phi_j - \phi_j), \mathcal{P}_{\geq 2j} g\rangle \\
&+ \langle C(\mathcal{T}_h\phi_j - \phi_j), \mathcal{P}_{\geq 2j} g\rangle + \langle (\mathcal{T}_h C - C)\phi_j, \mathcal{P}_{\geq 2j} g\rangle.
\end{aligned}
$$

For the last three terms in the right hand side of last equation, there are

$$
\begin{aligned}
&|\langle (\mathcal{T}_h C - C)\phi_j, \mathcal{P}_{\geq 2j} g\rangle| = \lambda_j |\langle \mathcal{T}_h\phi_j - \phi_j, \mathcal{P}_{\geq 2j} g\rangle| \leq \lambda_j \|\mathcal{T}_h\phi_j - \phi_j\|_\infty \|\mathcal{P}_{\geq 2j} g\|_1 \\
\lesssim &\lambda_j h^2 j^c j^{1/2} \lesssim h^2 j^{c+1/2-a}, \\
&|\langle C(\mathcal{T}_h\phi_j - \phi_j), \mathcal{P}_{\geq 2j} g\rangle| = \left| \sum_{k=2j}^{\infty} \langle C(\mathcal{T}_h\phi_j - \phi_j), \phi_k\rangle \langle \phi_k, g\rangle \right| \\
\leq &\sum_{k=2j}^{\infty} |\langle C(\mathcal{T}_h\phi_j - \phi_j), \phi_k\rangle| \|\phi_k\|_\infty \|g\|_1 \lesssim \sum_{k=2j}^{\infty} k^{-a} h^2 j^c \lesssim h^2 j^{c+1-a}
\end{aligned}
$$

and

$$
\begin{aligned}
&|\langle (\mathcal{T}_h C - C)(\mathcal{T}_h\phi_j - \phi_j), \mathcal{P}_{\geq 2j} g\rangle| \leq \|(\mathcal{T}_h C - C)(\mathcal{T}_h\phi_j - \phi_j)\|_\infty \|\mathcal{P}_{\geq 2j} g\|_1 \\
\lesssim &j^{1/2-a} \|\mathcal{T}_h\phi_j - \phi_j\|_2 \|\mathcal{P}_{\geq 2j} g\|_1 \lesssim j^{1/2-a} h^2 j^c j^{1/2} \lesssim h^2 j^{c+1-a}.
\end{aligned}
$$

The proof is complete by summing up the above. □

PROOF OF LEMMA S3 . For E_1,

$$
\begin{aligned}
E_1 = &\sup_{g\in\chi_1(\rho)} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{N_i(N_i-1)} \sum_{1\leq l_1\neq l_2\leq N_i} |\mathcal{T}_h\phi_j(T_{il_1})\mathcal{T}_h\mathcal{P}_{\geq 2j} g(T_{il_2})| |\delta_{il_1l_2}| \mathbf{1}_{(|\delta_{il_1l_2}|>A_n)} \\
\leq &\sup_{g\in\chi_1(\rho)} \frac{1}{n}\sum_{i=1}^{n} \frac{\|\mathcal{T}_h\phi_j\|_\infty \|\mathcal{T}_h\mathcal{P}_{\geq 2j} g\|_\infty}{N_i(N_i-1)} \sum_{1\leq l_1\neq l_2\leq N_i} |\delta_{il_1l_2}| \mathbf{1}_{(|\delta_{il_1l_2}|>A_n)} \\
\leq &\sup_{g\in\chi_1(\rho)} \|\mathcal{T}_h\phi_j\|_\infty \|\mathcal{T}_h\mathcal{P}_{\geq 2j} g\|_\infty \frac{Z_2}{A_n^{\alpha-1}} \lesssim \frac{Z_2}{hA_n^{\alpha-1}},
\end{aligned}
$$

where

$$
Z_2 := \frac{1}{n}\sum_{i=1}^{n} \frac{1}{N_i(N_i-1)} \sum_{1\leq l_1\neq l_2\leq N_i} |\delta_{il_1l_2}|^\alpha \text{ and } \mathbb{E}Z_2 = \mathbb{E}|\delta_{il_1l_2}|^\alpha \lesssim 1.
$$

For F_1,

$$
\begin{aligned}
\mathbb{E}|F_1| \leq &\sum_{g\in\chi_1(\rho)} \mathbb{E}\left[C_*^{>M}(\phi_j, \mathcal{P}_{\geq 2j} g)\right] \\
= &\sum_{g\in\chi_1(\rho)} \frac{\mathbb{E}\left[\tilde{A}_i(f, \mathcal{P}_{\geq 2j} g)\mathbf{1}_{(|\tilde{A}_i(f,\mathcal{P}_{\geq 2j} g)|>A_n\bar{N}_2^2\|f\|_\infty M)}\right]}{\bar{N}_2(\bar{N}_2-1)} \\
\leq &\sum_{g\in\chi_1(\rho)} \frac{\left[\mathbb{E}|\tilde{A}_i(f, \mathcal{P}_{\geq 2j} g)|^2 \mathbb{P}\left(|\tilde{A}_i(f, \mathcal{P}_{\geq 2j} g)| > A_n\bar{N}_2^2\|f\|_\infty M\right)\right]^{1/2}}{\bar{N}_2(\bar{N}_2-1)}
\end{aligned}
$$

$$\leq \sum_{g\in\chi_1(\rho)} \frac{\left[\mathbb{E}|\tilde{A}_i(\phi_j,\mathcal{P}_{\geq 2j}g)|^2\right]^{1/2}}{\bar{N}_2(\bar{N}_2-1)} \exp\left(-\frac{M\bar{N}_2}{4\|\mathcal{T}_h\mathcal{P}_{\geq 2j}g\|_\infty}\right)$$

$$\leq \frac{n^\rho M_2^{1/2}}{\bar{N}_2(\bar{N}_2-1)} \exp\left(-\frac{M\bar{N}_2}{4M_0}\right).$$

□

PROOF OF LEMMA S4. Recall that

$$\mathbb{E}|A_i(\phi_j,\mathcal{P}_{\geq 2j}g)|^2 \leq N_i^4 A_{i1}(\phi_j,\mathcal{P}_{\geq 2j}g) + 2N_i^3\{A_{i22}(\phi_j,\mathcal{P}_{\geq 2j}g) + A_{i23}(\phi_j,\mathcal{P}_{\geq 2j}g)\} + 2N_i^2 A_{i31}(\phi_j,\mathcal{P}_{\geq 2j}g).$$

with

$$\begin{aligned}
A_{i1}(\phi_j,\mathcal{P}_{\geq 2j}g) =& \mathbb{E}\left[|\langle X,\mathcal{T}_h\phi_j\rangle|^2|\langle X,\mathcal{T}_h\mathcal{P}_{\geq 2j}g\rangle|^2\right],\\
A_{i22}(\phi_j,\mathcal{P}_{\geq 2j}g) =& \mathbb{E}\left[|\langle X,\mathcal{T}_h\phi_j\rangle|^2(\|X\mathcal{T}_h\mathcal{P}_{\geq 2j}g\|_2^2+\sigma_X^2\|\mathcal{T}_h\mathcal{P}_{\geq 2j}g\|_2^2)\right],\\
A_{i23}(\phi_j,\mathcal{P}_{\geq 2j}g) =& \mathbb{E}\left[|\langle X,\mathcal{T}_h\mathcal{P}_{\geq 2j}g\rangle|^2(\|X\mathcal{T}_h\phi_j\|_2^2+\sigma_X^2\|\mathcal{T}_h\phi_j\|_2^2)\right],\\
A_{i31}(\phi_j,\mathcal{P}_{\geq 2j}g) =& \mathbb{E}\left[(\|X\mathcal{T}_h\phi_j\|_2^2+\sigma_X^2\|\mathcal{T}_h\phi_j\|_2^2)(\|X\mathcal{T}_h\mathcal{P}_{\geq 2j}g\|_2^2+\sigma_X^2\|\mathcal{T}_h\mathcal{P}_{\geq 2j}g\|_2^2)\right].
\end{aligned}$$

Note that $\mathbb{E}\|X\varphi\|_2^4 \leq \int_0^1 \mathbb{E}|X(t)|^4|\varphi(t)|^2dt\|\varphi\|_2^2 \lesssim \|\varphi\|_2^4$ and $\mathbb{E}(\|X\varphi\|_2^2+\sigma^2\|\varphi\|_2^2)^2 \lesssim \|\varphi\|_2^4$ for all $\varphi\in\mathcal{L}^2$, we have $A_{i31}(\phi_j,\mathcal{P}_{\geq 2j}g) \lesssim \|\mathcal{T}_h\phi_j\|_2^2\|\mathcal{T}_h\mathcal{P}_{\geq 2j}g\|_2^2$ and $A_{i22}(\phi_j,\mathcal{P}_{\geq 2j}g) \lesssim \left|\mathbb{E}\left[|\langle X,\mathcal{T}_h\phi_j\rangle|^4\right]\right|^{1/2}\|\mathcal{T}_h\mathcal{P}_{\geq 2j}g\|_2^2$. As $\|\mathcal{T}_h\phi_j\|_2 \leq \|\phi_j\|_2 \leq 1$, $\|\mathcal{T}_h\mathcal{P}_{\geq 2j}g\|_2 = \|\mathcal{T}_hg - \mathcal{T}_hP_{<2j}g\|_2 \leq 4h^{-1/2}\|g\|_1 + \|P_{<2j}g\|_2 \lesssim 4h^{-1/2}+(2j)^{1/2} \lesssim h^{-1/2}$ for all $g\in\chi_1(\rho)$ and $\mathbb{E}|\langle X,\mathcal{T}_h\phi_j\rangle|^4 \lesssim j^{-2a}$ by Lemma 2,

$$A_{i31}(\phi_j,\mathcal{P}_{\geq 2j}g) \lesssim h^{-1}, \quad A_{i22}(\phi_j,\mathcal{P}_{\geq 2j}g) \lesssim j^{-a}h^{-1}. \tag{S.43}$$

Next, we focus on quantity $\mathbb{E}\langle X,\mathcal{T}_h\mathcal{P}_{\geq 2j}g\rangle^4$ and note that

$$\langle X,\mathcal{T}_h\mathcal{P}_{\geq 2j}g\rangle = \langle\mathcal{T}_hX-\mathcal{P}_{<2j}X,g\rangle - \langle\mathcal{T}_hX-\mathcal{P}_{<2j}X,\mathcal{P}_{<2j}g\rangle. \tag{S.44}$$

For the second term in the right hand side of equation (S.44), note that

$$|\langle\mathcal{T}_hX-\mathcal{P}_{<2j}X,\mathcal{P}_{<2j}g\rangle| \leq \|\mathcal{T}_hX-\mathcal{P}_{<2j}X\|_2\|\mathcal{P}_{<2j}g\|_2 \lesssim j^{1/2}\|\mathcal{T}_hX-\mathcal{P}_{<2j}X\|_2,$$

and $\mathbb{E}[\|\mathcal{T}_hX-\mathcal{P}_{<2j}X\|_2^4] \lesssim j^{2-2a}$, we have

$$\mathbb{E}[|\langle\mathcal{T}_hX-\mathcal{P}_{<2j}X,\mathcal{P}_{<2j}g\rangle|^4] \lesssim j^{4-2a}. \tag{S.45}$$

For the first term in the right hand side of equation (S.44), we further have

$$\langle\mathcal{T}_hX-\mathcal{P}_{<2j}X,g\rangle = \langle\mathcal{T}_h\mathcal{P}_{<2j}X-\mathcal{P}_{<2j}X,g\rangle + \langle\mathcal{T}_h\mathcal{P}_{\geq 2j}X,g\rangle. \tag{S.46}$$

Note that

$$\begin{aligned}
&\left(\mathbb{E}\|\mathcal{T}_h\mathcal{P}_{<2j}X-\mathcal{P}_{<2j}X\|_\infty^4\right)^{\frac14}\\
\lesssim& \left\{\mathbb{E}\left(\sum_{k=1}^{2j-1}|\xi_k|\|\mathcal{T}_h\phi_k-\phi_k\|_\infty\right)^4\right\}^{\frac14}\\
\lesssim& \left\{\mathbb{E}\left(\sum_{k=1}^{2j-1}|\xi_k|h^2k^c\right)^4\right\}^{\frac14} \lesssim \sum_{k=1}^{2j-1}\left(\mathbb{E}\xi_k^4\right)^{\frac14}h^2k^c \lesssim \sum_{k=1}^{2j-1}\lambda_k^{\frac12}h^2k^c\\
\lesssim& \sum_{k=1}^{2j-1}h^2k^c \lesssim h^2j^{c+1} \lesssim j^{1-a}(h^4j^{2c+2a})^{1/2} \lesssim j^{1-a}.
\end{aligned} \tag{S.47}$$

For the second term in equation (S.46), as $\mathcal{T}_h g \geq 0$ and $\|\mathcal{T}_h g\|_1 = 1$ for all $g \in \chi_1(\rho)$,

$$
\begin{aligned}
&\mathbb{E}|\langle \mathcal{T}_h \mathcal{P}_{\geq 2j} X, g\rangle|^4 = \mathbb{E}|\langle \mathcal{P}_{\geq 2j} X, \mathcal{T}_h g\rangle|^4 \leq \mathbb{E}\langle |\mathcal{P}_{\geq 2j} X|^4, \mathcal{T}_h g\rangle \\
&= \int_0^1 \mathbb{E}|\mathcal{P}_{\geq 2j} X(t)|^4 \mathcal{T}_h g(t) dt \lesssim \int_0^1 j^{4-2a} \mathcal{T}_h g(t) dt \lesssim j^{4-2a}.
\end{aligned} \tag{S.48}
$$

Combine equation (S.44) to (S.48), we have $\mathbb{E}|\langle X, \mathcal{T}_h \mathcal{P}_{\geq 2j} g\rangle|^4 \lesssim j^{4-2a}$. Thus,

$$
\begin{aligned}
A_{i11}(\phi_j, \mathcal{P}_{\geq 2j} g) \leq& \left(\mathbb{E}|\langle X, \mathcal{T}_h \phi_j\rangle|^4 \mathbb{E}|\langle X, \mathcal{T}_h \mathcal{P}_{\geq 2j} g\rangle|^4\right)^{\frac{1}{2}} \\
\lesssim& (j^{-2a} j^{4-2a})^{\frac{1}{2}} \lesssim j^{2-2a}; \\
A_{i23}(\phi_j, \mathcal{P}_{\geq 2j} g) \leq& \|\mathcal{T}_h \phi_j\|_\infty^2 \mathbb{E}\left[|\langle X, \mathcal{T}_h \mathcal{P}_{\geq 2j} g\rangle|^2 (\|X\|_{L^2}^2 + \sigma^2)\right] \\
\lesssim& \left(\mathbb{E}|\langle X, \mathcal{T}_h \mathcal{P}_{\geq 2j} g\rangle|^4\right)^{1/2} \lesssim j^{2-a}.
\end{aligned} \tag{S.49}
$$

The proof is complete by combing equation (S.43) and (S.49) and the arbitrariness of g.

□

PROOF OF LEMMA S5. Denote $\xi_{ik} = \langle X_i, \phi_k\rangle$, then $X_i = \sum_{k=1}^\infty \xi_{ik}\phi_k$. By Minkowski Inequality and Assumption 1,

$$
\left(\mathbb{E}\|X_i\|^4\right)^{\frac{1}{2}} = \left\{\mathbb{E}\left(\sum_{k=1}^\infty \xi_{ik}^2\right)^2\right\}^{\frac{1}{2}} \leq \sum_{k=1}^\infty \left(\mathbb{E}\xi_{ik}^4\right)^{\frac{1}{2}} \leq \sum_{k=1}^\infty C\lambda_k < \infty.
$$

For all $u \in [h, 1-h]$,

$$
\begin{aligned}
&\left|\mathcal{T}_h \frac{\phi_k}{f}(u) - \frac{\phi_k}{f}(u)\right| = \left|\frac{1}{h}\int \mathrm{K}\left(\frac{u-t}{h}\right)\frac{\phi_k(t)}{f(t)}\mathrm{d}t - \frac{\phi_k(u)}{f(u)}\right| \\
=&\left|\int_{-1}^1 \mathrm{K}(v)\left\{\frac{\phi_k(u)}{f(u)} - hv\left(\frac{\phi_k(u)}{f(u)}\right)^{(1)}(u) + \frac{h^2v^2}{2}\left(\frac{\phi_k(u)}{f(u)}\right)^{(2)}(u^*)\right\}\mathrm{d}v - \frac{\phi_k(u)}{f(u)}\right| \\
\lesssim& h^2|\phi_k^{(2)}(t)|_\infty \lesssim h^2 k^c.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathbb{E}\left[\left\langle X_i f, \mathcal{T}_h\frac{\phi_k}{f}\right\rangle^4\right] = \mathbb{E}\left[\left\{\int_0^1 X_i(u) f(u)\frac{1}{h}\int \mathrm{K}\left(\frac{u-y}{h}\right)\frac{\phi_k(y)}{f(y)}\mathrm{d}y\mathrm{d}u\right\}^4\right] \\
=&\mathbb{E}\left[\left\{\int_h^{1-h} X_i(u) f(u)\frac{1}{h}\int \mathrm{K}\left(\frac{u-y}{h}\right)\frac{\phi_k(y)}{f(y)}\mathrm{d}y\mathrm{d}u\right.\right. \\
&\left.\left.+\left(\int_0^h + \int_{1-h}^1\right) X_i(u) f(u)\frac{1}{h}\int \mathrm{K}\left(\frac{u-y}{h}\right)\frac{\phi_k(y)}{f(y)}\mathrm{d}y\mathrm{d}u\right\}^4\right] \\
\lesssim&\mathbb{E}[(\xi_{ik} + h\|X_i\| + h^2k^c\|X_i\|)^4] \\
\lesssim&\mathbb{E}\{\xi_{ik}^4 + h^4\mathbb{E}\|X_i\|^4 + (h^4k^{2c})^2\mathbb{E}\|X_i\|^4\} \\
\lesssim& k^{-2a},
\end{aligned} \tag{S.50}
$$

where the last inequality is due to Assumption 1 and $h^4 j^{2a+2c} = O(1)$, $hj^a \log n = O(1)$.

By similar analysis, one has

$$\sum_{k>j}\left\langle X_i f, \mathcal{T}_h \frac{\phi_k}{f}\right\rangle^2 \lesssim \sum_{k>j}\left\langle \mathcal{T}_h X_i f, \frac{\phi_k}{f}\right\rangle^2 + h\|X_i\|^2 \text{ and } \left\|\frac{1}{f}\mathcal{T}_h X_i f - \mathcal{T}_h X_i\right\|^2 \lesssim h\|X_i\|^2.$$

Thus,

$$\begin{aligned}\sum_{k>j}\left\langle X_i f, \mathcal{T}_h \frac{\phi_k}{f}\right\rangle^2 &\lesssim \sum_{k>j}\left\langle \frac{1}{f}\mathcal{T}_h X_i f - \sum_{r=1}^{j}\xi_{ir}\phi_r, \phi_k\right\rangle^2 + h\|X_i\|^2 \\ &\lesssim \left\|\mathcal{T}_h \sum_{r=1}^{j}\xi_{ir}\phi_r - \sum_{r=1}^{j}\xi_{ir}\phi_r\right\|^2 + \left\|\sum_{r=j+1}^{\infty}\xi_{ir}\phi_r\right\|^2 + h\|X_i\|^2\end{aligned} \tag{S.51}$$

By Minkowski inequality,

$$\left(\mathbb{E}\left\|\sum_{r=j+1}^{\infty}\xi_{ir}\phi_r\right\|^4\right)^{\frac{1}{2}} = \left\{\mathbb{E}\left(\sum_{r>j}\xi_{ir}^2\right)^2\right\}^{\frac{1}{2}} \le \sum_{r>j}\left(\mathbb{E}\xi_{ir}^4\right)^{\frac{1}{2}} \lesssim \sum_{r>j}\lambda_r \lesssim j^{1-a}. \tag{S.52}$$

By similar argument as in (S.50), one has $\|\mathcal{T}_h\phi_k - \phi_k\| \lesssim h + h^2k^c$. Thus,

$$\left\|\mathcal{T}_h \sum_{r=1}^{j}\xi_{ir}\phi_r - \sum_{r=1}^{j}\xi_{ir}\phi_r\right\| \le \sum_{r=1}^{j}|\xi_{ir}|\|\mathcal{T}_h\phi_r - \phi_r\| \lesssim \sum_{r=1}^{j}|\xi_{ir}|(h^2r^c + h)$$

and then

$$\begin{aligned}\left(\mathbb{E}\left\|\mathcal{T}_h \sum_{r=1}^{j}\xi_{ir}\phi_r - \sum_{r=1}^{j}\xi_{ir}\phi_r\right\|^4\right)^{\frac{1}{4}} &\lesssim \sum_{r=1}^{j}\left(\mathbb{E}\xi_{ir}^4\right)^{\frac{1}{4}}(h^2r^c + h) \\ \lesssim \sum_{r=1}^{j}\lambda_r^{\frac{1}{2}}(h^2r^c+h) &\lesssim \sum_{r=1}^{j}(h^2r^c+h) \lesssim h^2j^{c+1} + hj \lesssim j^{1-a},\end{aligned} \tag{S.53}$$

where the last inequality is by $h^4j^{2a+2c} \lesssim 1$ and $hj^a\log n \lesssim 1$.

Combine equation (S.51), (S.52) and (S.53), the proof is complete by

$$\begin{aligned}&\mathbb{E}\left(\sum_{k>j}\left\langle X_i f, \mathcal{T}_h\frac{\phi_k}{f}\right\rangle^2\right)^2 \\ \lesssim &\mathbb{E}\left\|\mathcal{T}_h \sum_{r=1}^{j}\xi_{ir}\phi_r - \sum_{r=1}^{j}\xi_{ir}\phi_r\right\|^4 + \mathbb{E}\left\|\sum_{r=j+1}^{\infty}\xi_{ir}\phi_r\right\|^4 + h^2\mathbb{E}\|X_i\|^4 \lesssim j^{2-2a}.\end{aligned}$$

□

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