TALK: SAITO'S CONJECTURE ON CHARACTERISTIC CLASSES OF CONSTRUCTIBLE ÉTALE SHEAVES

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1. Introduction

This talk is based on joint work with Yigeng Zhao.

1.1. For vector bundles on varieties, we have Chern/Characteristic classes. Chern classes measures non-triviality of vector bundles. Before 1966, Grothendieck conjectured that there exists a theory of characteristic classes for constructible étale sheaves and a discrete Riemann-Roch type formula (see [Récoltes et Semailles, Note 87_1]). Such construction requires a generalization of Artin-Serre-Swan type local invariants to higher dimensional varieties. Let us fix a few notation.

- k: perfect field.
- $\Lambda = \mathbb{F}_\ell, \mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$ for a prime $\ell \in k^\times$.
- X: variety over k.
- $\mathcal{F}$: constructible etale sheaf of Λ-modules on X.

1.2. What is a constructible etale sheaf? The most interesting example comes from the following case: there is an open subscheme $U \subseteq X$, and $\mathcal{F}$ determines a Λ-representation of the etale fundamental group $\pi_1(U)$. When $\mathcal{F}$ comes from a representation of $\pi_1(U)$, then we say $\mathcal{F}$ is a locally constant (smooth) sheaf on X. Otherwise $\mathcal{F}$ has ramification along the boundary $X\backslash U$. Its characteristic class $cc_{X/k}(\mathcal{F}) \in CH_0(X)$ (or its refined version: the Swan class $\mathrm{Sw}^{\mathrm{cc}}_{X/k}(\mathcal{F}) \in CH_0(X\backslash U)$) measures the ramification of $\mathcal{F}$ along the boundary $X\backslash U$. In some sense, the characteristic/Swan class measures the "distance" between $\mathcal{F}$ and the smooth sheaf $\Lambda^{\oplus \mathrm{rank}\mathcal{F}}$ (measures the non-smoothness of $\mathcal{F}$).

Example 1.3. Assume that X is connected, smooth and proper of dimension d over k. When $\mathcal{F}$ is smooth on X, then we have the Gauss-Bonnet-Chern formula for the Euler-Poincare characteristic:

$$\chi(X_{\bar{k}}, \mathcal{F}) = \chi(X_{\bar{k}}, \Lambda^{\oplus \mathrm{rank}\mathcal{F}}) = \mathrm{rank}\mathcal{F} \cdot \chi(X_{\bar{k}}, \Lambda) = \mathrm{rank}\mathcal{F} \cdot \mathrm{deg} c_d(\Omega^{1,\vee}_{X/k}). \tag{1.3.1}$$

In general, if $\mathcal{F}$ is smooth on U, then $\chi(X_{\bar{k}}, \mathcal{F}) - \chi(X_{\bar{k}}, \Lambda^{\oplus \mathrm{rank}\mathcal{F}})$ is the degree of a zero cycle class supported on the boundary $X\backslash U$ (namely, the Swan classes):

$$\chi(X_{\bar{k}}, \mathcal{F}) - \chi(X_{\bar{k}}, \Lambda^{\oplus \mathrm{rank}\mathcal{F}}) = -\mathrm{deg}(\mathrm{Sw}^{\mathrm{cc}}_{X/k}(\mathcal{F})). \tag{1.3.2}$$

If moreover X is a smooth proper curve, we have the well-known Grothendieck-Ogg-Safarevich formula

$$\chi(X_{\bar{k}}, \mathcal{F}) - \chi(X_{\bar{k}}, \Lambda^{\oplus \mathrm{rank}\mathcal{F}}) = -\sum_{x \in |X\backslash U|} a_x(\mathcal{F}), \tag{1.3.3}$$

where $a_x(\mathcal{F}) = \mathrm{dim}\mathcal{F}_{\bar{\eta}_x} - \mathrm{dim}\mathcal{F}_{\bar{x}} + \mathrm{Sw}_x(\mathcal{F})$ is the Artin conductor of $\mathcal{F}$ at x, $\mathrm{Sw}_x(\mathcal{F})$ is the Swan conductor.

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1.4. Assume that X is smooth and connected over k. Up to now, there are two kinds of characteristic classes ($C_{X/k}$ and $cc_{X/k}$) and three kinds of Swan classes ($\mathrm{Sw}^{as}_{X/k}$, $\mathrm{Sw}^{cc}_{X/k}$ and $\mathrm{Sw}^{ks}_{X/k}$).

(1) The cohomological characteristic class $C_{X/k}(\mathcal{F}) \in H^0(X, \mathcal{K}_{X/k})$ is implicitly defined in [SGA5] and studied by Abbes and Saito around 2007. (See also Kashiwara-Schapira's book "Sheaves on manifolds")
(2) The geometric characteristic class $cc_{X/k}(\mathcal{F}) \in CH_0(X)$ is defined by Saito around 2015.

Even though their definitions and constructions are very different, Saito conjectures that they are essentially the same.

Conjecture 1.5 (Takeshi Saito, [Sai17]). *Consider the cycle class map* $\mathrm{cl} : CH_0(X) \to H^0(X, \mathcal{K}_{X/k})$, *where* $\mathcal{K}_{X/k} = Rf^!\Lambda$ *and* $f : X \to \mathrm{Spec}k$. *For any constructible étale sheaf* $\mathcal{F}$ *on* X, *we have*

$$\mathrm{cl}(cc_{X/k}(\mathcal{F})) = C_{X/k}(\mathcal{F}).$$

Please refer to [UYZ20] for the version of Swan classes. Note that, when $k = \mathbb{F}_p$ is a finite field and $\Lambda = \mathbb{Z}/\ell^m$ and if X is projective and smooth, then we have $H^0(X, \mathcal{K}_{X/k}) \simeq H^1(X, \mathbb{Z}/\ell^m)^\vee \simeq \pi_1^{\mathrm{ab}}(X)/\ell^m$, which may highly non-trivial.

Here is our main result:

Theorem 1.6 (Y-Zhao, [YZ25]). *Saito's conjecture holds if* X *is quasi-projective.*

If using more ∞-category, we could be able to prove Saito's conjecture in general.

2. Idea of the proof

In the following, we omit to write R or L to denote the derived functors.

2.1. Before describing the idea of proofs, let me discuss a little bit about $\mathcal{F}$-smooth morphisms (or $\mathcal{F}$-ULA morphisms). This is a cohomological version of the usual smooth morphisms. Let $\mathcal{F}$ be a constructible étale sheaf on X. In general, for a separated morphism $f : X \to S$ of finite type, we say f is $\mathcal{F}$-smooth if the relative purity holds for any base change diagram

$$\begin{array}{ccc} W & \xrightarrow{i} & X \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\delta} & S, \end{array} \tag{2.1.1}$$

i.e., the canonical morphism

$$i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \xrightarrow{c_{\delta,f,\mathcal{F}}} i^!\mathcal{F} \tag{2.1.2}$$

is an isomorphism. The map (2.1.2) is defined to be the composition

$$i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \xrightarrow{id\otimes \mathrm{b.c}} i^*\mathcal{F} \otimes^L i^!f^*\Lambda \xrightarrow{\mathrm{adj}} i^!i_!(i^*\mathcal{F} \otimes^L i^!f^*\Lambda) \xrightarrow[\simeq]{\mathrm{proj.formula}} i^!(\mathcal{F} \otimes^L i_!i^!f^*\Lambda) \xrightarrow{\mathrm{adj}} i^!\mathcal{F}.$$

Example 2.2. (1) If $f : X \to S$ is a smooth morphism, then f is Λ-smooth for the constant sheaf Λ.
(2) If $f = \mathrm{id}_X : X \to X$ is the identity, then f is $\mathcal{F}$-smooth if and only if $\mathcal{F}$ is smooth (locally constant) on X.
(3) If $S = \mathrm{Spec}k$ is a point, then $X \to \mathrm{Spec}k$ is $\mathcal{F}$-smooth for any constructible etale sheaf $\mathcal{F}$.

Definition 2.3. For ($\mathcal{F}$,$X \xrightarrow{f} S$), its NA-locus (non-acyclicity locus) is the smallest closed subset $Z \subseteq X$ such that $X \backslash Z \to S$ is $\mathcal{F}$-smooth.

2.4. Now let me explain our ideas how to prove Theorem 1.6. We use fibration method.

2.4.1. *Wonderful case.* If there is a $\mathcal{F}$-smooth morphism $f : X \to Y$ to a smooth curve, then we proved that $C_{X/k}(\mathcal{F})$ is determined by the family $\{C_{X_v/v}(\mathcal{F}|_{X_v})\}_{v\in|Y|}$. The later family is encoded by the relative cohomological characteristic class $C_{X/Y}(\mathcal{F}) \in H^0(X, \mathcal{K}_{X/Y})$ with $\mathcal{K}_{X/Y} = Rf^!\Lambda$, which is introduced in [YZ21] under transversal conditions and generated to ULA-conditions by Lu and Zheng.

2.4.2. *Good fibration.* In general, we don't have such $\mathcal{F}$-smooth fibration. But not too bad, after blowing-up, we could find a good Lefschetz pencil by a result of Saito-Yatagawa: The morphism $f : X \to Y$ is a good fibration with respect to $\mathcal{F}$ if f is $\mathcal{F}$-smooth outside finitely many closed points such that each fiber contains at most one point of the NA-locus.

In this case, we still have $C_{X/Y}(\mathcal{F})$ (encoding the information $\{C_{X_v/v}(\mathcal{F}|_{X_v})\}_{v\in|Y|}$). But this family cannot determine $C_{X/k}(\mathcal{F})$ anymore. But by the wonderful case, the obstruction comes from a class supported on the NA-locus. Thus we have to construct a class $C_\Delta(\mathcal{F})$ supported on the NA-locus, which is called the (cohomological) non-acyclicity class. This NA-class $C_\Delta(\mathcal{F})$ satisfies the fibration formula below. Similar formula also holds for the geometric characteristic class $cc_{X/k}(\mathcal{F})$.

In order to compare $C_{X/k}(\mathcal{F})$ with $cc_{X/k}(\mathcal{F})$, we only need to calculate $C_\Delta(\mathcal{F})$ for isolated singularities. This is given by the cohomological Milnor formula.

Now, we have a new class: NA-class. You can run the previous argument and then get a family/relative version of this NA-class. In the proof of cohomological Milnor formula, we need this relative version to do deformation!

3. Non-acyclicity classes

3.1. We recall the transversality condition introduced in [YZ25, 2.1], which is a relative version of the transversality condition studied by Saito [Sai17, Definition 8.5]. Consider the following cartesian diagram in Sch_S:

(3.1.1)
$$\begin{array}{ccc} W & \xrightarrow{i} & X \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\delta} & Y. \end{array}$$

By [YZ25, 2.11], there is a functor $\delta^\Delta : D_{\mathrm{ctf}}(X, \Lambda) \to D_{\mathrm{ctf}}(W, \Lambda)$ such that for any $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$, we have a distinguished triangle

(3.1.2)
$$i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \xrightarrow{c_{\delta,f,\mathcal{F}}} i^!\mathcal{F} \to \delta^\Delta\mathcal{F} \xrightarrow{+1} .$$

If $\delta^\Delta(\mathcal{F})$=0, then we say that the morphism δ is $\mathcal{F}$-*transversal*.

3.2. Consider a commutative diagram in Sch_S:

(3.2.1)
$$\begin{array}{ccccc} Z & \xhookrightarrow{\tau} & X & \xrightarrow{f} & Y, \\ & & & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & & & S & \end{array}$$

where $\tau : Z \to X$ is a closed immersion and g is a smooth morphism. Let us denote the diagram (3.2.1) simply by $\Delta = \Delta^Z_{X/Y/S}$ Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $X\backslash Z \to Y$ is $\mathcal{F}|_{X\backslash Z}$-smooth and that $h : X \to S$ is $\mathcal{F}$-smooth.

3.3. Let $i : X \times_Y X \to X \times_S X$ be the base change of the diagonal morphism $\delta : Y \to Y \times_S Y$:

$$
\begin{array}{ccc}
X & = & X \\
\downarrow{\scriptstyle \delta_1} & \square & \downarrow{\scriptstyle \delta_0} \\
X \times_Y X & \xrightarrow{i} & X \times_S X \\
\downarrow{\scriptstyle p} & \square & \downarrow{\scriptstyle f\times f} \\
Y & \xrightarrow{\delta} & Y \times_S Y,
\end{array}
\qquad (f : X \to Y)
\tag{3.3.1}
$$

where δ_0 and δ_1 are the diagonal morphisms. Put $K_{X/S} = h^!\Lambda$ and $\mathcal{K}_\Delta := \delta^\Delta \mathcal{K}_{X/S} \simeq \delta_1^* \delta^\Delta \delta_{0*} \mathcal{K}_{X/S}$. We have the following distinguished triangle

$$\mathcal{K}_{X/Y} \to \mathcal{K}_{X/S} \to \mathcal{K}_\Delta \xrightarrow{+1} . \tag{3.3.2}$$

We put

$$\mathcal{H}_S := R\mathcal{H}om_{X\times_S X}(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^! \mathcal{F}) \xleftarrow{\simeq} \mathcal{T}_S := \mathcal{F} \boxtimes_S^L D_{X/S}(\mathcal{F}).$$

We have the following microlocal result:

Lemma 3.4. *$\delta_1^* \delta^\Delta \mathcal{T}_S$ is supported on Z.*

Definition 3.5 ([YZ25, Definition 4.6])**.** The relative cohomological characteristic class $C_{X/S}(\mathcal{F})$ is the composition (cf. [YZ25, 3.1])

$$\Lambda \xrightarrow{\mathrm{id}} R\mathcal{H}om(\mathcal{F}, \mathcal{F}) \underset{\simeq}{\to} \delta_0^! \mathcal{H}_S \underset{\simeq}{\leftarrow} \delta_0^! \mathcal{T}_S \to \delta_0^* \mathcal{T}_S \xrightarrow{\mathrm{ev}} \mathcal{K}_{X/S}. \tag{3.5.1}$$

The non-acyclicity class $C_\Delta(\mathcal{F}) \in H^0_Z(X, \mathcal{K}_\Delta)$ is the composition

$$\Lambda \to \delta_0^! \mathcal{H}_S \xleftarrow{\simeq} \delta_0^! \mathcal{T}_S \simeq \delta_1^! i^! \mathcal{T}_S \to \delta_1^* i^! \mathcal{T}_S \to \delta_1^* \delta^\Delta \mathcal{T}_S \xleftarrow{\simeq} \tau_* \tau^! \delta_1^* \delta^\Delta \mathcal{T}_S \to \tau_* \tau^! \mathcal{K}_{X/Y/S}. \tag{3.5.2}$$

If the following condition holds:

$$H^0(Z, \mathcal{K}_{Z/Y}) = 0 \text{ and } H^1(Z, \mathcal{K}_{Z/Y}) = 0 \tag{3.5.3}$$

then the map $H^0_Z(X, \mathcal{K}_{X/S}) \to H^0_Z(X, \mathcal{K}_{X/Y/S})$ is an isomorphism. In this case, the class $C_\Delta(\mathcal{F}) \in H^0_Z(X, \mathcal{K}_{X/Y/S})$ defines an element of $H^0_Z(X, \mathcal{K}_{X/S})$.

Now we summarize the functorial properties for the non-acyclicity classes (cf. [YZ25, Theorem 1.9, Proposition 1.11, Theorem 1.12, Theorem 1.14]).

Theorem 3.6 (Y-Zhao,[YZ25])**.**

(1) (Fibration formula) If $H^0(Z, \mathcal{K}_{Z/Y}) = H^1(Z, \mathcal{K}_{Z/Y}) = 0$, then we have

$$C_{X/S}(\mathcal{F}) = c_r(f^* \Omega^{1,\vee}_{Y/S}) \cap C_{X/Y}(\mathcal{F}) + C_\Delta(\mathcal{F}) \quad \text{in} \quad H^0(X, \mathcal{K}_{X/S}). \tag{3.6.1}$$

(2) (Pull-back) Let $b : S' \to S$ be a morphism of Noetherian schemes. Let $\Delta' = \Delta^{Z'}_{X'/Y'/S'}$ be the base change of $\Delta = \Delta^Z_{X/Y/S}$ by $b : S' \to S$. Let $b_X : X' = X \times_S S' \to X$ be the base change of b by $X \to S$. Then we have

$$b_X^* C_\Delta(\mathcal{F}) = C_{\Delta'}(b_X^* \mathcal{F}) \quad \text{in} \quad H^0_{Z'}(X', \mathcal{K}_{X'/Y'/S'}), \tag{3.6.2}$$

where $b_X^* : H^0_Z(X, \mathcal{K}_{X/Y/S}) \to H^0_{Z'}(X', \mathcal{K}_{\Delta'})$ is the induced pull-back morphism.

(3) (Proper push-forward) Consider a diagram $\Delta' = \Delta^{Z'}_{X'/Y/S}$. Let $s : X \to X'$ be a proper morphism over Y such that $Z \subseteq s^{-1}(Z')$. Then we have

$$s_*(C_\Delta(\mathcal{F})) = C_{\Delta'}(Rs_* \mathcal{F}) \quad \text{in} \quad H^0_{Z'}(X', \mathcal{K}_{X'/Y/S}), \tag{3.6.3}$$

where $s_* : H^0_Z(X, \mathcal{K}_\Delta) \to H^0_{Z'}(X', \Delta')$ is the induced push-forward morphism.

(4) (Cohomological Milnor formula) Assume $S = \operatorname{Spec} k$. If $Z = \{x\}$ and Y is a smooth curve, then we have

$$C_\Delta(\mathcal{F}) = -\operatorname{dimtot} R\Phi_{\bar{x}}(\mathcal{F}, f) \quad \text{in} \quad \Lambda = H^0_x(X, \mathcal{K}_{X/k}), \tag{3.6.4}$$

where $R\Phi(\mathcal{F}, f)$ is the complex of vanishing cycles and $\operatorname{dimtot} = \dim + \operatorname{Sw}$ is the total dimension.

(5) (Cohomological conductor formula) Assume $S = \operatorname{Spec} k$. If Y is a smooth connected curve over k and $Z = f^{-1}(y)$ for a closed point $y \in |Y|$, then we have

$$f_* C_\Delta(\mathcal{F}) = -a_y(Rf_*\mathcal{F}) \quad \text{in} \quad \Lambda = H^0_y(Y, \mathcal{K}_{Y/k}). \tag{3.6.5}$$

(6) The formation of non-acyclicity classes is also compatible with specialization maps (cf. [YZ25, Proposition 4.17]).

3.7. Let X be a smooth connected curve over k. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ and $Z \subseteq X$ be a finite set of closed points such that $\mathcal{F}|_{X\backslash Z}$ are smooth. By the cohomological Milnor formula (3.6.4), we have the following (motivic) expression for the Artin conductor of $\mathcal{F}$ at $x \in Z$

$$a_x(\mathcal{F}) = \operatorname{dimtot} R\Phi_{\bar{x}}(\mathcal{F}, \mathrm{id}) = -C^{\{x\}}_{U/U/k}(\mathcal{F}|_U), \tag{3.7.1}$$

where U is any open subscheme of X such that $U \cap Z = \{x\}$. By (3.6.1), we get the following cohomological Grothendieck-Ogg-Shafarevich formula (cf. [YZ25, Corollary 6.6]):

$$C_{X/k}(\mathcal{F}) = \operatorname{rank}\mathcal{F} \cdot c_1(\Omega^{1,\vee}_{X/k}) - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \quad \text{in} \quad H^0(X, \mathcal{K}_{X/k}). \tag{3.7.2}$$

3.8. **Idea of the proof.** May assume $Y = \mathbb{A}^1$. Consider

$$\begin{array}{ccccc} Z \times \mathbb{P}^1 & \overset{\tau}{\hookrightarrow} & X \times \mathbb{P}^1 & \xrightarrow{f \times \mathrm{id}} & Y \times \mathbb{P}^1, \\ & & & {\scriptstyle ft}\searrow \quad \swarrow & \\ & & & \mathbb{P}^1 & \end{array} \tag{3.8.1}$$

and $\mathcal{G} = \mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{L}_!(ft)$, where $\mathcal{L}$ is the Artin-Schreier sheaf on $\mathbb{A}^1$ associated with some character $\psi : \mathbb{F}_p \to \Lambda^*$. After taking a finite extension $\mathbb{P} \to \mathbb{P}^1$, we may assume $\mathcal{G} \in D^b_c(\Delta \times \mathbb{P}\backslash\infty)$. Applying the pull-back and specialization formulas to $C_{\Delta \times \mathbb{P}\backslash\infty}(\mathcal{G}) \in H^0(Z \times \mathbb{P}, \mathcal{K}_{Z \times \mathbb{P}/\mathbb{P}}) = \bigoplus_{x \in Z} \Lambda$, we get

$$C_\Delta(\Psi_{\mathrm{pr}_2}(\mathcal{G})) = C_\Delta(\mathcal{F}).$$

Since $\Psi_{\mathrm{pr}_2}(\mathcal{G})$ is supported on Z, by definition of NA class, we get

$$C_\Delta(\mathcal{F}) = C_\Delta(\Psi_{\mathrm{pr}_2}(\mathcal{G})) = -\sum_{x \in Z} \operatorname{dimtot} R\Phi_{\bar{x}}(\mathcal{F}, f) \cdot [x].$$

Remark 3.9. I found an open question due to Drinfeld in Beilinson's paper [Bei07]: For microlocal-analysis, our habitat is a smooth variety, which does not look very natural for the story. What intrinsic geometry is truly relevant for the micro-local analysis of sheaves? It should make sense outside the smooth context, so that one could play with singular spaces directly, without embedding them into smooth ones.

Here is a partial answer:

Smooth case	Singular case
Characteristic cycle	relative cohomological class and NA class

References

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