清华大学报告

Date. 2025.03.20

Characteristic classes of constructible étale sheaves

Joint with Yigeng Zhao

§1 Introduction

For vector bundles on varieties, we have Chern/Characteristic classes

Grothendieck: How to define discrete characteristic classes ? for constructible étale sheaves

Riemann-Roch type formula?

What is a constructible étale sheaf?

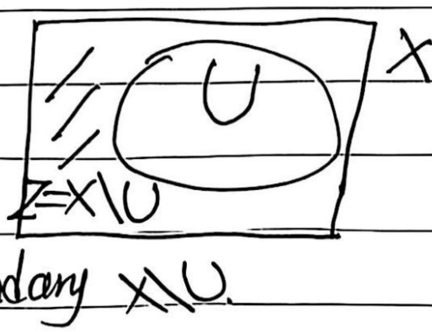
We make the following simplification: view them as ℓ-adic rep of the fundamental group of an open subschemes.

k: perfect field. X/k: variety. $\Lambda = \mathbb{F}_\ell, \mathbb{Q}_\ell, \overline{\mathbb{Q}}_\ell$ (ℓ invertible on k)

$\mathcal{F} \in D_{ctf}(X,\Lambda)$ = Derived cat of constructible étale sheaves on X (with finite tor-dim)

简化: $\exists$ open $U \subseteq X$, and $\mathcal{F}|_U$ defines an ~~Λ~~ Λ rep of $\pi_1^{et}(U)$.

When $\mathcal{F}$ comes from a rep of $\pi_1^{et}(X)$, then say $\mathcal{F}$ is locally constant/smooth on X.



Otherwise, $\mathcal{F}$ has ramification along the boundary $X \setminus U$.

The characteristic class of $\mathcal{F}$, denoted by $cc_{X/k}(\mathcal{F})$, in some sense measures the distance ~~of~~ between $\mathcal{F}$ and smooth sheaves.

Example (1) When X smooth and projective, then the Euler-Poincaré characteristic $\chi(X_{\bar{k}},\mathcal{F}) = \deg CC_{X/k}(\mathcal{F})$.

— When $\mathcal{F}$ is smooth, then

$$\chi(X_{\bar{k}},\mathcal{F}) = \mathrm{rank}\mathcal{F}\cdot\chi(X_{\bar{k}},\Lambda) \overset{\text{Gauss-Bonnet-Chern}}{=\!=\!=} \mathrm{rank}\mathcal{F}\cdot\deg \underset{\substack{\uparrow \\ C_{\dim X}(\Omega^{1,\vee}_{X/k})}}{C_{\dim X}(\cdots}$$

In general, $\chi(X_{\bar{k}},\mathcal{F}) - \mathrm{rank}\mathcal{F}\cdot\chi(X_{\bar{k}},\Lambda) = \deg\left(\begin{smallmatrix}\text{Swan class}\\ \text{supported on } X\setminus U\end{smallmatrix}\right)$

(2) If X is a smooth proper curve, ~~when~~ we have Grothendieck-Ogg-Safarev formula $\chi(X_{\bar{k}},\mathcal{F}) - \mathrm{rank}\mathcal{F}\cdot\chi(X_{\bar{k}},\Lambda) = -\sum_{x\in|X\setminus U|} \underset{\substack{\uparrow\\ \text{Artin conductor}}}{a_x(\mathcal{F})}$.

(3) k: finite field, we have global ε-factor $\varepsilon(X,\mathcal{F}) = \det(-\mathrm{Frob}_k; R\Gamma(X_{\bar{k}},\mathcal{F}))^{-1}$.

$$\cdot\ \frac{\varepsilon(X,\mathcal{F})}{\varepsilon(X,\Lambda)^{\mathrm{rank}\mathcal{F}}} \overset{\text{猜测}}{=} \text{“}p\text{-th roots of unity”}\cdot\text{“generalized Jacobi sum”}\cdot p^{-\chi(X_{\bar{k}},\mathcal{F})}\cdot\langle\mathcal{F}, CC_{X/k}\mathcal{F}\rangle.$$

The Euler-Poincaré characteristic $\chi(X_{\bar{k}},\mathcal{F})$ and $\varepsilon(X,\mathcal{F})$ are two important invariants in Grothendieck L-function $L(X,\mathcal{F},t)$:

$$L(X,\mathcal{F},t) = \prod_{x\in|X|}\left(1-t^{\deg(x)}\mathrm{Frob}_x; \mathcal{F}_{\bar{x}}\right)^{-1}$$

$$= \det(1-t\,\mathrm{Frob}_k; R\Gamma(X_{\bar{k}},\mathcal{F}))^{-1}$$

$$\overset{\text{Functional equation (类比 Riemann-Zeta)}}{=\!=\!=} \varepsilon(X,\mathcal{F})\cdot t^{-\chi(X_{\bar{k}},\mathcal{F})}\cdot L(X, D(\mathcal{F}), t^{-1}).$$

where $D(\mathcal{F}) = R\mathcal{H}om(\mathcal{F}, \mathcal{K}_{X/k})$ Verdier dual of $\mathcal{F}$.

Conjecture (Kato-Saito, 2008, twist formula) For any smooth sheaf $\mathcal{G}$ on X, we have $\varepsilon(X,\mathcal{F}\otimes\mathcal{G}) = \varepsilon(X,\mathcal{F})^{\mathrm{rk}\mathcal{G}}\cdot\det\mathcal{G}(\rho_X(-CC_{X/k}\mathcal{F}))$,

where $\rho_X: CH_0(X)\to\pi_1^{ab}(X)$ is the reciprocity map in higher dimensional class field thm.

$$\pi_1^{ab}(X)\xrightarrow{\det\mathcal{G}}\Lambda^\times$$

Thm A (Umezaki-Y-Zhao) Kato-Saito's conjecture is true.

Up to now, there are three kinds of characteristic classes:

Kato–Saito, 2004/2008, Swan classes $\mathrm{Sw}\mathcal{F} \in CH_0(\bar{X}\setminus X)_{\mathbb{Q}}$

Abbes–Saito, 2007 Cohomological characteristic class

$Z^{1,v}_{X/k})$ $C_{X/k}(\mathcal{F}) \in H^0(X, \mathcal{K}_{X/k})$

Saito 2016 geometric characteristic class

$CC_{X/k}(\mathcal{F}) \in CH_0(X)$.

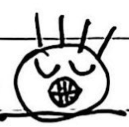
The constructions of these three classes are very different.

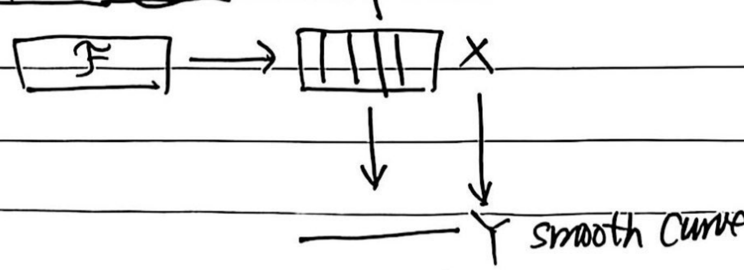
<u>Conjecture (Saito 2016)</u> Under the cycle class map $CH_0(X) \xrightarrow{cl} H^0(X, \mathcal{K}_{X/k})$, they are essentially the same!

Note that, when $k = \mathbb{F}_p$ is finite, and $\Lambda = \mathbb{Z}/\ell^m$, X proj smooth, then we have $H^0(X, \mathcal{K}_{X/k}) \cong H^1(X, \mathbb{Z}/\ell^m)^\vee \cong \pi_1^{ab}(X)/\ell^m$.

<u>Thm B (Y–Zhao)</u> If X is quasi-projective, then Saito's conjecture is true.

<u>Ideas of proving Thm A + Thm B</u> We use fibration method, here is a sketch.

<u>Wonderful case</u>  If there is a smooth fibration



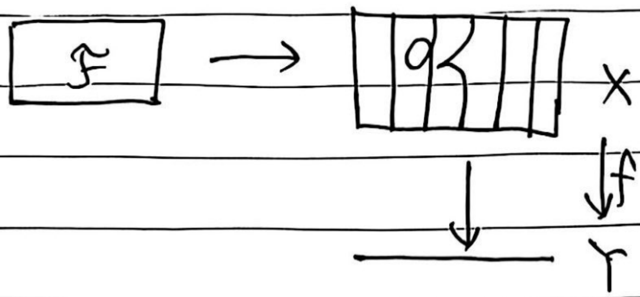
then $C_{X/k}(\mathcal{F})$ is determined by $\{C_{X_v/v}(\mathcal{F}|X_v)\}_{v\in|Y|}$, which define a family version of characteristic classes.

We introduced relative characteristic class $C_{X/Y}(\mathcal{F})$

$<$ Y–Zhao
Lu–Zheng

In general, we don't have such $\mathcal{F}$-smooth fibration. But not too bad, after blowing-up, we could find a good Lefschetz pencil.

good fibration = $\mathcal{F}$-smooth fibration outside finitely many closed points



In this case, we still have $C_{X/Y}(\mathcal{F})$ [recording $\{C_{X_v/v}(\mathcal{F}|_{X_v})\}_{v\in|Y|}$]. But they cannot determine $C_{X/k}(\mathcal{F})$.
But by wonderful case, their differences comes from a class supported on the bad locus = non-$\mathcal{F}$-smooth locus = NA locus.

Thus we have to construct a class supported on NA locus

$$\text{NA class} = C_\Delta(\mathcal{F}) = C_{X/Y/k}(\mathcal{F}),$$

which measures their differences ($C_{X/k}(\mathcal{F})$ & $C_{X/Y}(\mathcal{F})$):

$$C_{X/k}(\mathcal{F}) = c_1(\Omega^{1,\vee}_{Y/k}) \cap C_{X/Y}(\mathcal{F}) + C_\Delta(\mathcal{F})$$

↑

Fibration formula.

Now we have a new class $C_\Delta(\mathcal{F})$. We can do previous construction again! ⇒ relative version of $C_\Delta(\mathcal{F})$

fibration formula for $C_\Delta(\mathcal{F})$

continue....

§2 NA Class

Fix a diagram $\Delta = \left(\begin{tikzcd} Z \arrow[r, hook] & X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & & S \end{tikzcd}\right)$ with g = smooth of rel. dim r.

For $\mathcal{F} \in D^b_c(\Delta)$, i.e., $\mathcal{F} \in D_{ctf}(X, \Lambda)$ s.t. h is $\mathcal{F}$-smooth
f is $\mathcal{F}$-smooth outside Z.

We have a new cohomology class, called NA-class of $\mathcal{F}$,

$$C_\Delta(\mathcal{F}) \in H^0_Z(X, \mathcal{K}_\Delta) \left(\xleftarrow{\text{when } Z \text{ small}} H^0(Z, \mathcal{K}_{Z/S}) \right)$$

$$H^0(Z, \mathcal{K}_{Z/Y}) = H^1(Z, \mathcal{K}_{Z/Y}) = 0$$

Theorem (Y–Zhao)

(1) Fibration formula: When Z small, we have

$$C_{X/S}(\mathcal{F}) = c_r(f^*\Omega^{1,\vee}_{Y/S}) \cap C_{X/Y}(\mathcal{F}) + C_\Delta(\mathcal{F})$$

(2) pull-back For any $S' \xrightarrow{b} S$, let $\Delta' = b^*\Delta = \Delta \times_S S'$, $\mathcal{F}' = b^*\mathcal{F}$,
then $b^* C_\Delta(\mathcal{F}) = C_{\Delta'}(\mathcal{F}')$.

(3) proper pushforward Consider $\Delta' \xrightarrow[\text{proper}]{s} \Delta$ of the form
$$\begin{tikzcd} Z \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & Z' \arrow[d, hook] \\ X \arrow[r, "s"] \arrow[dr] \arrow[ddr, bend right] & X' \arrow[d] \arrow[dd, bend left] \\ & Y \arrow[d] \\ & S \end{tikzcd}$$
then $s_* C_\Delta(\mathcal{F}) = C_{\Delta'}(Rs_*\mathcal{F})$

(4) Cohomological Milnor formula When $\Delta = \left(\begin{tikzcd} Z = \{x\} \arrow[r, hook] & X \arrow[r] \arrow[dr] & Y \arrow[d, "\text{smooth curve}"] \\ & & S = \mathrm{Spec}\,k \end{tikzcd}\right)$

then for $\mathcal{F} \in D^b_c(\Delta)$, we have $C_\Delta(\mathcal{F}) = -\mathrm{dimtot}\, R\Phi_f(\mathcal{F})_x$ in $H^0(\{x\}) = \Lambda$.

(5) Cohomological conductor formula ~~Conductor~~

Consider $\Delta = \left(\begin{tikzcd} Z = f^{-1}(y) \arrow[r] \arrow[d] & y \arrow[d] \\ X \arrow[r, "f", "\text{proper}"'] \arrow[dr] & Y \arrow[d, "\text{smooth curve}"] \\ & S \end{tikzcd}\right)$

$S = \mathrm{Spec}\,k$

Apply (3) to $\Delta \xrightarrow{f} \Delta'$, where $\Delta' = \left(\begin{tikzcd} \{y\} \arrow[r, hook] & Y = Y \arrow[d] \\ & \mathrm{Spec}\,k \end{tikzcd}\right)$

get $f_* C_\Delta(\mathcal{F}) = C_{\Delta'}(Rf_*\mathcal{F})$
$\overset{(4)}{=} -a_y(Rf_*\mathcal{F})$.

Date.

(6) Specialization of $C_\Lambda(\mathcal{F})$

(7) Cohomological GOS formula: Special case of (4) and (1).

$$\begin{tikzcd} Z \arrow[r, hook] & X \arrow[r, equal] \arrow[dr] & X \arrow[d, "\text{smooth curve}"] \\ & & \operatorname{Spec} k \end{tikzcd}$$

$$C_{X/k}(\mathcal{F}) = \operatorname{rank}\mathcal{F}\cdot C_1(\Omega^{1,\vee}_{X/k}) - \sum_{x\in Z} a_x(\mathcal{F})\cdot[x] \text{ in } H^0(X, \mathcal{K}_{X/k}).$$

Construction of NA classes

We will omit "R" for derived functors. Everything is in the derived sense.

Consider $(*)$
$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "p"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] & \\ W \arrow[r, hook, "\delta"', "\text{closed immersion}"'] & T & T\setminus W \arrow[l, hook', "j"'] \end{tikzcd}$$

We have a "pull-back" functor $\delta^\Delta = i^!(-\otimes f^* j_*\Lambda) : D_{ctf}(Y) \to D_{ctf}(X)$.
such that for any $\mathcal{F}\in D_{ctf}(Y,\Lambda)$, there is a distinguished triangle
$$i^*\mathcal{F}\otimes p^*\delta^!\Lambda \longrightarrow i^!\mathcal{F} \longrightarrow \delta^\Delta\mathcal{F} \xrightarrow{+}$$
The first map is induced by
$$i_!(i^*\mathcal{F}\otimes p^*\delta^!\Lambda) \cong \mathcal{F}\otimes i_! p^*\delta^!\Lambda \cong \mathcal{F}\otimes f^*\delta_!\delta^!\Lambda \xrightarrow{\text{adj}} \mathcal{F}.$$

We say δ is $\mathcal{F}$-transversal if $\delta^\Delta(\mathcal{F}) = 0$.

Say $Y\xrightarrow{f} T$ is $\mathcal{F}$-smooth / $\mathcal{F}$-ULA if for all $(*)$, δ is $\mathcal{F}$-transversal.

Example If $\mathcal{F}=\Lambda$ and f is a smooth morphism, then f is $\mathcal{F}$-smooth (cohomological smooth) Peter Scholze

NA locus of $(Y\xrightarrow{f}T, \mathcal{F})$ = smallest closed subset $Z\subseteq Y$ such that $Y\setminus Z\xrightarrow{f} T$ is $\mathcal{F}$-smooth.

Date.

As before, fix $\Delta = \Delta^{Z}_{X/Y/S} = \left(\begin{tikzcd} Z \arrow[r, hook, "\tau"] & X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & & S \end{tikzcd} \right)$ with g smooth.

Consider the following diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\delta_1"] \arrow[dd, bend right=60, "f"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\delta_0"] \\ X\times_Y X \arrow[r, hook, "i"] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & X\times_S X \arrow[d, "f\times f"] \\ Y \arrow[r, hook, "\delta"'] & Y\times_S Y \end{tikzcd}$$

$K_{X/S} = h^!\Lambda$

$K_\Delta = \delta^\Delta K_{X/S}$

— We have a distinguished triangle

$K_{X/Y} \longrightarrow K_{X/S} \longrightarrow K_\Delta \xrightarrow{+}$

where the first map is induced by the Chern class of $\Omega_{Y/S}$.

<u>Lemma</u> For $\mathcal{F} \in D^b_c(\Delta)$, i.e., h is $\mathcal{F}$-smooth, f is $\mathcal{F}$-smooth outside Z.

(1) $R\mathcal{H}om_{X\times_S X}(pr_2^*\mathcal{F}, pr_1^!\mathcal{F}) \underset{\text{Lu-Zheng}}{\cong} \mathcal{F} \boxtimes^L_S D_{X/S}(\mathcal{F}) =: \mathcal{H}_S$

(2) $\delta_1^*\delta^\Delta \mathcal{H}_S$ is supported on Z.

<u>Definition</u> (1) The relative cohomological characteristic class $C_{X/S}(\mathcal{F})$ ($\in H^0(X, K_{X/S})$) is the composition

$$\Lambda \xrightarrow{id} R\mathcal{H}om(\mathcal{F},\mathcal{F}) \cong \delta_0^!\mathcal{H}_S \longrightarrow \delta_0^*\mathcal{H}_S \xrightarrow{ev} K_{X/S}.$$

(2) The non-acyclicity class $C_\Delta(\mathcal{F}) \in H^0_Z(X, K_\Delta)$ is the composition

$$\Lambda \longrightarrow \delta_0^!\mathcal{H}_S = \delta_1^! i^! \mathcal{H}_S \longrightarrow \delta_1^*\delta^\Delta \mathcal{H}_S \overset{\text{Lemma (2)}}{\cong} \tau_*\tau^!\delta_1^*\delta^\Delta \mathcal{H}_S$$
$$\longrightarrow \tau_*\tau^!\delta^\Delta\delta_1^*\mathcal{H}_S$$
$$\xrightarrow{ev} \tau_*\tau^!\delta^\Delta K_{X/S} = \tau_*\tau^! K_\Delta.$$

Say Z is small if $H^0(Z, K_{Z/Y}) = H^1(Z, K_{Z/Y}) = 0$, in this case, $H^0_Z(X, K_{X/S}) \cong H^0(Z, K_{Z/Y})$ and $C_\Delta(\mathcal{F})$ is a class in $H^0(Z, K_{Z/Y})$.

Date.

§3 Proofs

For the fibration formula, we need a little bit ∞-category. We can construct a coherent commutative diagram between nine diagrams (九图)

$$
\begin{tikzcd}
 & T_*K_{Z/Y} \arrow[rr] \arrow[dd] & & K_{X/Y} \arrow[rr] \arrow[dd] & & j_*K_{U/Y} \arrow[dd] \\
\Lambda[-1] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[rr] \arrow[dd] & & \Lambda \arrow[ur, "c_{U/Y}"] \arrow[ul, dashed] \arrow[dd, equal] & \\
 & T_*K_{Z/S} \arrow[rr] \arrow[dd] & & K_{X/S} \arrow[rr] \arrow[dd] & & j_*K_{U/S} \arrow[dd] \\
0 \arrow[ur] \arrow[rr] \arrow[dd] & & \Lambda \arrow[ur, "c_{X/S}"] \arrow[rr, equal] \arrow[dd, equal] & & \Lambda \arrow[ur, "c_{U/S}"] \arrow[dd] & \\
 & T_*T^!K_{X/Y/S} \arrow[rr] & & K_{X/Y/S} = K_\Delta \arrow[rr] & & j_*K_{U/Y/S} \\
\Lambda \arrow[ur, "c^Z_{X/Y/S}"] \arrow[uuur, dashed] \arrow[rr, equal] & & \Lambda \arrow[ur, "c_{X/Y/S}"] \arrow[rr] & & 0 \arrow[ur] &
\end{tikzcd}
$$

By assumption $\mathrm{Hom}(\Lambda, T_*K_{Z/Y}) = \mathrm{Hom}(\Lambda, \Sigma T_*K_{Z/Y}) = 0$

We have a unique lifting $\Lambda \xrightarrow{c^Z_{X/Y/S}} T_*K_{Z/S}$ of NA class

$\Lambda \xrightarrow{c_{X/Y}} K_{X/Y}$ of $\Lambda \xrightarrow{c_{U/Y}} j_*K_{U/Y}$.

We have a commutative diagram

$$
\begin{tikzcd}
\Lambda \arrow[r] \arrow[d, "{(c^Z_{X/Y/S},\, c_{X/Y}) = \beta}"'] \arrow[dr, phantom, "\circlearrowleft"] & \Lambda \oplus \Lambda \oplus \Lambda \arrow[d, "{(c^Z_{X/Y/S},\, c_{X/S},\, c_{U/Y})}"] \\
Z \arrow[r] \arrow[d, equal] & T_*T^!K_{X/Y/S} \oplus K_{X/S} \oplus j_*K_{U/Y} \\
\mathrm{cofib}(T_*K_{Z/Y} \to T_*K_{Z/S} \oplus K_{X/Y}) &
\end{tikzcd}
$$

$\Rightarrow c_{X/S} = \delta^! c_{X/Y} + c^Z_{X/Y/S}$ in $H^0(X, K_{X/S})$.

<u>For cohomological Milnor formula</u>

$$\begin{tikzcd} Z\times\mathbb{P}^1 \arrow[r, hook, "\tau"] & X\times\mathbb{P}^1 \arrow[r, "f\times id"] \arrow[dr, "ft"'] & Y\times\mathbb{P}^1 \arrow[d] \\ & & \mathbb{P}^1 \end{tikzcd}$$

May assume $Y=\mathbb{A}^1$ and consider

Put $\mathcal{G}=pr_1^*\mathcal{F}\otimes\mathcal{L}_\psi(ft)$, where $\mathcal{L}$ is the Artin-Schreier sheaf on $\mathbb{A}^1$ associated to some character $\psi:\mathbb{F}_p\to\Lambda^\times$.

After taking a finite ext $\mathbb{P}\to\mathbb{P}^1$, we may assume $\mathcal{G}\in D_c^b(\Delta\times\mathbb{P}\setminus\{\infty\})$

$$C_{\Delta\times\mathbb{P}\setminus\infty}(\mathcal{G})\in H^0(Z\times\mathbb{P}\setminus\{\infty\},\ K_{Z\times\mathbb{P}\setminus\{\infty\}/\mathbb{P}})=\bigoplus_{x\in Z}\Lambda$$

specialization to ∞ — pull-back res_0

$$C_\Delta(\underbrace{\Psi_{pr_2}(\mathcal{G})}_{\text{supported on } Z}) \qquad\qquad C_\Delta(\mathcal{F})$$

$\|$ ← Laumon

$$-\sum_{x\in Z}\operatorname{dimtot} R\Phi_x(\mathcal{F},f)\cdot[x]$$

<u>Blow-up formula</u> X, Y: smooth connected

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\pi: \text{blow-up of } X \text{ along } Y"] \\ Y \arrow[r, hook, "i"', "\operatorname{codim}=r\geq 1"] & X \end{tikzcd}$$

Then $\pi_* C_{\tilde{X}/k}(\pi^*\mathcal{F})=C_{X/k}(\mathcal{F})+(r-1)\,i_*C_{Y/k}(i^*\mathcal{F})$ in $H^0(X, K_{X/k})$

Pf) use $R\pi_*\pi^*\mathcal{F}\cong\mathcal{F}\oplus\bigoplus_{t=1}^{r-1} i_*i^*\mathcal{F}(-t)[-2t]$.

<u>Proof of Saito's conjecture</u> Induction on n.

$d=0$ OK. $C_{X/k}=CC_{X/k}=\text{rank}$.

$d=1$ cohomological GOS formula (fibration formula) ~~and~~

Suppose $d\geq 1$, taking a finite ext of k (of order prime to ℓ), ~~may assume there~~ and a blow-up $X'\to X$, may assume there is a good fibration $X\xrightarrow{f}\mathbb{P}^1$ w.r.t. $SS\mathcal{F}$, i.e., f is $\mathcal{F}$-smooth outside a finite set $\{x_v\}_{v\in\Sigma}$, $\underset{\text{finite}}{\Sigma\subseteq|Y|}$, for $v\neq u$, $x_v\neq x_u$

Date.

$\Omega^1_{k(\mathbb{P}^1)/k}$

For a ~~meromorphic~~ rational 1-form ω on $\mathbb{P}^1$, write $c_1(\Omega^{1}_{\mathbb{P}^1}) = \sum_{v\in|\mathbb{P}^1|} \mathrm{ord}_v(\omega)\cdot[v]$

Then we have

$$CC_{X/k}(\mathcal{F}) = -\sum_{v\in|Y\setminus\Sigma|} \mathrm{ord}_v(\omega)\cdot CC_{X_v}(\mathcal{F}|_{X_v}) - \sum_{v\in\Sigma} \mathrm{dimtot}\, R\Phi_{\bar{x}_v}(\mathcal{F}_i \cdot [x_v]$$

Same for $C_{X/k}(\mathcal{F})$.

By induction hypothesis and cohomological Milnor formula, we get

$$C_{X/k}(\mathcal{F}) = cl(CC_{X/k}(\mathcal{F})).$$

$\blacksquare$