LECTURE ON SAITO'S CONJECTURE ON CHARACTERISTIC CLASSES OF CONSTRUCTIBLE ÉTALE SHEAVES

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ABSTRACT. This talk is based on joint work with Yigeng Zhao.

1. Introduction

1.1. Let us first introduce a few notation and discuss some motivations.

- k: perfect field of characteristic $p > 0$.
- $\Lambda = \mathbb{F}_\ell$: finite field of characteristic $\ell \neq p$.
- X: smooth scheme over k.
- $\mathcal{F}$: constructible etale sheaf of Λ-modules on X. (simply viewed as a Λ-representation of the etale fundamental group $\pi_1(U)$ for an open subscheme $U \subseteq X$)
- Geometric ramification studies the behavior of $\mathcal{F}$ along the boundary $X \backslash U$.
- The characteristic class of $\mathcal{F}$ measures the ramification of $\mathcal{F}$. (It is the discrete version of the characteristic class for a vector bundle.)
- For any separated morphism $f : X \to Y$, we put $\mathcal{K}_{X/Y} = Rf^!\Lambda$ and $D_{X/Y}(-) = R\mathcal{H}om(-, \mathcal{K}_{X/Y})$
- We omit to write R or L to denote the derived functors.

There are two kinds of characteristic classes. Their definitions are quite different.

Conjecture 1.2 (Takeshi Saito, 2015)**.** *Consider the cycle class map* $\mathrm{cl} : CH_0(X) \to H^0(X, \mathcal{K}_{X/k})$, *where* $\mathcal{K}_{X/k} = Rf^!\Lambda$ *and* $f : X \to \mathrm{Spec}k$*. Then we have*

$$\mathrm{cl}(cc_{X/k}(\mathcal{F})) = C_{X/k}(\mathcal{F}).$$

- The cohomological characteristic class $C_{X/k}(\mathcal{F}) \in H^0(X, \mathcal{K}_{X/k})$ is implicitly defined in [SGA7] and studied by Abbes and Saito around 2007.
- The geometric characteristic class $cc_{X/k}(\mathcal{F}) \in CH_0(X)$ is defined by Saito around 2015.
- They can be viewed as higher dimensional (global) analogues of the Artin conductors (local invariants).
- Characteristic classes are quite important! Here is an application. Assume k is a finite field and X smooth and projective. Consider the Grothendieck L-function

  $$L(X, \mathcal{F}, t) = \det(1 - \mathrm{Frob} \cdot t; R\Gamma(X_{\bar{k}}, \mathcal{F}))^{-1}.$$

  It satisfies the following functional equation

  $$L(X, \mathcal{F}, t) = t^{-\chi(X,\mathcal{F})} \cdot \varepsilon(X, \mathcal{F}) \cdot L(X, D(\mathcal{F}), t^{-1}).$$

  Then we have the global index formula for the Euler-Poincare characteristic

  $$\chi(X, \mathcal{F}) = \deg cc_{X/k}(\mathcal{F}) = \mathrm{Tr} C_{X/k}(\mathcal{F}),$$

  and the twist formula for the global epsilon factor

  $$\varepsilon(X, \mathcal{F} \otimes \mathcal{G}) = \varepsilon(X, \mathcal{F})^{\mathrm{rk}\mathcal{G}} \cdot \det\mathcal{G}(\rho_X(-cc_{X/k}(\mathcal{F}))),$$

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where $\rho_X : CH_0(X) \to \pi_1^{\mathrm{ab}}(X)$ is the reciprocity map and $\mathcal{G}$ is any smooth sheaf on X.

Here is our main result:

Theorem 1.3 (Yang-Zhao, 2022)**.** *Saito's conjecture holds if X is quasi-projective.*

If using more ∞-category, we could be able to prove Saito's conjecture in general.

1.4. **Idea of the proof.** In some sense, we have to give a cohomological construction for Saito's characteristic cycle. So, we have to propose a cohomological way to study ramification theory.

2. Cohomological approach

2.1. We recall the transversality condition introduced in [3, 2.1], which is a relative version of the transversality condition studied by Saito [1, Definition 8.5]. Consider the following cartesian diagram in Sch_S:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "p"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ W \arrow[r, "\delta"] & T. \end{tikzcd} \tag{2.1.1}$$

By [3, 2.11], there is a functor $\delta^\Delta : D_{\mathrm{ctf}}(Y,\Lambda) \to D_{\mathrm{ctf}}(X,\Lambda)$ such that for any $\mathcal{F} \in D_{\mathrm{ctf}}(Y,\Lambda)$, we have a distinguished triangle

$$i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \xrightarrow{c_{\delta,f,\mathcal{F}}} i^!\mathcal{F} \to \delta^\Delta\mathcal{F} \xrightarrow{+1} . \tag{2.1.2}$$

The first map is defined to be the composition

$$i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \xrightarrow{id\otimes \mathrm{b.c}} i^*\mathcal{F} \otimes^L i^!f^*\Lambda \xrightarrow{\mathrm{adj}} i^!i_!(i^*\mathcal{F} \otimes^L i^!f^*\Lambda) \xrightarrow[\simeq]{\mathrm{proj.formula}} i^!(\mathcal{F} \otimes^L i_!i^!f^*\Lambda) \xrightarrow{\mathrm{adj}} i^!\mathcal{F}.$$

We say that the morphism δ is $\mathcal{F}$-*transversal* if $\delta^\Delta(\mathcal{F})$=0.

The following definition can be viewed as a cohomological version of smooth morphisms (cf. Lu-Zheng and Peter Scholze).

Definition 2.2. Fix $\mathcal{F} \in D_{\mathrm{ctf}}(Y,\Lambda)$. We say f is $\mathcal{F}$-smooth if for any such diagram (2.1.1), the morphism δ is $\mathcal{F}$-transversal.

2.3. Consider a commutative diagram in Sch_S:

$$\begin{tikzcd} Z \arrow[r, hook, "\tau"] & X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y, \arrow[dl, "g"] \\ & & S & \end{tikzcd} \tag{2.3.1}$$

where $\tau : Z \to X$ is a closed immersion and g is a smooth morphism. Let us denote the diagram (2.3.1) simply by $\Delta = \Delta^Z_{X/Y/S}$ Let $\mathcal{F} \in D_{\mathrm{ctf}}(X,\Lambda)$ such that $X\backslash Z \to Y$ is $\mathcal{F}|_{X\backslash Z}$-smooth and that $h : X \to S$ is $\mathcal{F}$-smooth.

2.4. Let $i : X \times_Y X \to X \times_S X$ be the base change of the diagonal morphism $\delta : Y \to Y \times_S Y$:

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, hook, "\delta_1"] \arrow[dd, bend right=60, "f"'] \arrow[dr, phantom, "\square"] & X \arrow[d, hook, "\delta_0"] \\
X \times_Y X \arrow[r, "i"] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & X \times_S X \arrow[d, "f \times f"] \\
Y \arrow[r, "\delta"] & Y \times_S Y,
\end{tikzcd}
\tag{2.4.1}
$$

where δ_0 and δ_1 are the diagonal morphisms. Put $K_{X/S} = h^!\Lambda$ and $\mathcal{K}_\Delta := \delta^\Delta \mathcal{K}_{X/S} \simeq \delta_1^* \delta^\Delta \delta_{0*} \mathcal{K}_{X/S}$. We have the following distinguished triangle

$$\mathcal{K}_{X/Y} \to \mathcal{K}_{X/S} \to \mathcal{K}_\Delta \xrightarrow{+1} . \tag{2.4.2}$$

We put

$$\mathcal{H}_S := R\mathcal{H}om_{X \times_S X}(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^! \mathcal{F}) \xleftarrow{\simeq} \mathcal{T}_S := \mathcal{F} \boxtimes_S^L D_{X/S}(\mathcal{F}).$$

Lemma 2.5. *$\delta_1^* \delta^\Delta \mathcal{T}_S$ is supported on Z.*

Definition 2.6 ([3, Definition 4.6])**.** The relative cohomological characteristic class $C_{X/S}(\mathcal{F})$ is the composition (cf. [3, 3.1])

$$\Lambda \xrightarrow{\mathrm{id}} R\mathcal{H}om(\mathcal{F}, \mathcal{F}) \underset{\simeq}{\to} \delta_0^! \mathcal{H}_S \underset{\simeq}{\leftarrow} \delta_0^! \mathcal{T}_S \to \delta_0^* \mathcal{T}_S \xrightarrow{\mathrm{ev}} \mathcal{K}_{X/S}. \tag{2.6.1}$$

The non-acyclicity class $C_\Delta(\mathcal{F}) \in H_Z^0(X, \mathcal{K}_\Delta)$ is the composition

$$\Lambda \to \delta_0^! \mathcal{H}_S \xleftarrow{\simeq} \delta_0^! \mathcal{T}_S \simeq \delta_1^! i^! \mathcal{T}_S \to \delta_1^* i^! \mathcal{T}_S \to \delta_1^* \delta^\Delta \mathcal{T}_S \xleftarrow{\simeq} \tau_* \tau^! \delta_1^* \delta^\Delta \mathcal{T}_S \to \tau_* \tau^! \mathcal{K}_{X/Y/S}. \tag{2.6.2}$$

If the following condition holds:

$$H^0(Z, \mathcal{K}_{Z/Y}) = 0 \text{ and } H^1(Z, \mathcal{K}_{Z/Y}) = 0 \tag{2.6.3}$$

then the map $H_Z^0(X, \mathcal{K}_{X/S}) \to H_Z^0(X, \mathcal{K}_{X/Y/S})$ is an isomorphism. In this case, the class $C_\Delta(\mathcal{F}) \in H_Z^0(X, \mathcal{K}_{X/Y/S})$ defines an element of $H_Z^0(X, \mathcal{K}_{X/S})$.

Now we summarize the functorial properties for the non-acyclicity classes (cf. [3, Theorem 1.9, Proposition 1.11, Theorem 1.12, Theorem 1.14]).

Theorem 2.7 (Yang-Zhao)**.**

(1) (Fibration formula) If $H^0(Z, \mathcal{K}_{Z/Y}) = H^1(Z, \mathcal{K}_{Z/Y}) = 0$, then we have

$$C_{X/S}(\mathcal{F}) = c_r(f^* \Omega^{1,\vee}_{Y/S}) \cap C_{X/Y}(\mathcal{F}) + C_\Delta(\mathcal{F}) \quad \text{in} \quad H^0(X, \mathcal{K}_{X/S}). \tag{2.7.1}$$

(2) (Pull-back) Let $b : S' \to S$ be a morphism of Noetherian schemes. Let $\Delta' = \Delta^{Z'}_{X'/Y'/S'}$ be the base change of $\Delta = \Delta^Z_{X/Y/S}$ by $b : S' \to S$. Let $b_X : X' = X \times_S S' \to X$ be the base change of b by $X \to S$. Then we have

$$b_X^* C_\Delta(\mathcal{F}) = C_{\Delta'}(b_X^* \mathcal{F}) \quad \text{in} \quad H^0_{Z'}(X', \mathcal{K}_{X'/Y'/S'}), \tag{2.7.2}$$

where $b_X^* : H_Z^0(X, \mathcal{K}_{X/Y/S}) \to H^0_{Z'}(X', \mathcal{K}_{\Delta'})$ is the induced pull-back morphism.

(3) (Proper push-forward) Consider a diagram $\Delta' = \Delta^{Z'}_{X'/Y/S}$. Let $s : X \to X'$ be a proper morphism over Y such that $Z \subseteq s^{-1}(Z')$. Then we have

$$s_*(C_\Delta(\mathcal{F})) = C_{\Delta'}(Rs_* \mathcal{F}) \quad \text{in} \quad H^0_{Z'}(X', \mathcal{K}_{X'/Y/S}), \tag{2.7.3}$$

where $s_* : H_Z^0(X, \mathcal{K}_\Delta) \to H^0_{Z'}(X', \Delta')$ is the induced push-forward morphism.

(4) (Cohomological Milnor formula) Assume $S = \mathrm{Spec}k$. If $Z = \{x\}$ and Y is a smooth curve, then we have

$$C_\Delta(\mathcal{F}) = -\mathrm{dimtot}R\Phi_{\bar{x}}(\mathcal{F}, f) \quad \text{in} \quad \Lambda = H^0_x(X, \mathcal{K}_{X/k}), \tag{2.7.4}$$

where $R\Phi(\mathcal{F}, f)$ is the complex of vanishing cycles and $\mathrm{dimtot} = \dim + \mathrm{Sw}$ is the total dimension.

(5) (Cohomological conductor formula) Assume $S = \mathrm{Spec}k$. If Y is a smooth connected curve over k and $Z = f^{-1}(y)$ for a closed point $y \in |Y|$, then we have

$$f_* C_\Delta(\mathcal{F}) = -a_y(Rf_*\mathcal{F}) \quad \text{in} \quad \Lambda = H^0_y(Y, \mathcal{K}_{Y/k}), \tag{2.7.5}$$

where $a_y(\mathcal{G}) = \mathrm{rank}\mathcal{G}|_{\bar{\eta}} - \mathrm{rank}\mathcal{G}_{\bar{y}} + \mathrm{Sw}_y\mathcal{G}$ is the Artin conductor of the object $\mathcal{G} \in D_{\mathrm{ctf}}(Y, \Lambda)$ at y and η is the generic point of Y.

(6) The formation of non-acyclicity classes is also compatible with specialization maps (cf. [3, Proposition 4.17]). We call (2.7.1) the fibration formula for characteristic class, which is motivated from [2].

2.8. Let X be a smooth connected curve over k. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ and $Z \subseteq X$ be a finite set of closed points such that the cohomology sheaves of $\mathcal{F}|_{X\backslash Z}$ are locally constant. By the cohomological Milnor formula (2.7.4), we have the following (motivic) expression for the Artin conductor of $\mathcal{F}$ at $x \in Z$

$$a_x(\mathcal{F}) = \mathrm{dimtot}R\Phi_{\bar{x}}(\mathcal{F}, \mathrm{id}) = -C^{\{x\}}_{U/U/k}(\mathcal{F}|_U), \tag{2.8.1}$$

where U is any open subscheme of X such that $U \cap Z = \{x\}$. By (2.7.1), we get the following cohomological Grothendieck-Ogg-Shafarevich formula (cf. [3, Corollary 6.6]):

$$C_{X/k}(\mathcal{F}) = \mathrm{rank}\mathcal{F} \cdot c_1(\Omega^{1,\vee}_{X/k}) - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \quad \text{in} \quad H^0(X, \mathcal{K}_{X/k}). \tag{2.8.2}$$

2.9. **Idea of the proof.** May assume $Y = \mathbb{A}^1$. Consider

$$\begin{tikzcd} Z \times \mathbb{P}^1 \arrow[r, hook, "\tau"] & X \times \mathbb{P}^1 \arrow[rr, "f \times \mathrm{id}"] \arrow[dr, "ft"'] & & Y \times \mathbb{P}^1, \arrow[dl] \\ & & \mathbb{P}^1 & \end{tikzcd} \tag{2.9.1}$$

and $\mathcal{G} = \mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{L}_!(ft)$, where $\mathcal{L}$ is the Artin-Schreier sheaf on $\mathbb{A}^1$ associated with some character $\psi : \mathbb{F}_p \to \Lambda^*$. After taking a finite extension $\mathbb{P} \to \mathbb{P}^1$, we may assume $\mathcal{G} \in D^b_c(\Delta \times \mathbb{P}\backslash\infty)$. Applying the pull-back and specialization formulas to $C_{\Delta\times\mathbb{P}\backslash\infty}(\mathcal{G}) \in H^0(Z \times \mathbb{P}, \mathcal{K}_{Z\times\mathbb{P}/\mathbb{P}}) = \bigoplus_{x\in Z}\Lambda$, we get

$$C_\Delta(\Psi_{\mathrm{pr}_2}(\mathcal{G})) = C_\Delta(\mathcal{F}).$$

Since $\Psi_{\mathrm{pr}_2}(\mathcal{G})$ is supported on Z, by definition of NA class, we get

$$C_\Delta(\mathcal{F}) = C_\Delta(\Psi_{\mathrm{pr}_2}(\mathcal{G})) = -\sum_{x\in Z} \mathrm{dimtot}R\Phi_{\bar{x}}(\mathcal{F}, f) \cdot [x].$$

References

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[3] E. Yang and Y. Zhao, *Cohomological Milnor formula and Saito's conjecture on characteristic classes*, 2022, arXiv:2209.11086. ↑2, ↑3, ↑4