LECTURE ON NON-ACYCLICITY CLASSES

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Abstract. In this lecture, we introduce two classes supported on the non-acyclicity locus of a separated morphism relatively to a constructible sheaf. One is defined in a cohomological way by using localized categorical trace, another is constructed via geometric method by using Saito's characteristic cycle. As applications of these two classes,

(1) We prove cohomological analogs of the Milnor formula and the conductor formula for constructible sheaves on (not necessarily smooth) varieties.
(2) We propose a (relative version of) Milnor type formula for non-isolated singularities.

This talk is based on joint work with Jiangnan Xiong and Yigeng Zhao.

Contents

1. Introduction

1.1. Let k be a perfect field of characteristic $p > 0$ and $S = \mathrm{Spec}k$. Let Λ be a finite field of characteristic $\ell \neq p$. Let X be a smooth scheme over S and $f : X \to Y$ a flat morphism of finite type to a smooth curve Y over S. If f has an isolated singularity at a closed point $x_0 \in |X|$, there is an invariant $\mu(X/Y, x_0)$ supported on x_0, called the Milnor number. The Milnor formula [4, Théorème 2.4] proved by Deligne says that the Milnor number is related to the total dimension at x_0 of the vanishing cycles $R\Phi(f, \Lambda)$ of f for the constant sheaf Λ, i.e.,

$$(-1)^n \mu(X/Y, x_0) = -\mathrm{dimtot} R\Phi_{\bar{x}_0}(f, \Lambda), \tag{1.1.1}$$

where $n = \mathrm{dim}X$ and $\mathrm{dimtot} = \mathrm{dim} + \mathrm{Sw}$ denotes the total dimension. Later in [5], Deligne proposed a Milnor formula for any constructible sheaf $\mathcal{F}$ of Λ-modules on X, which is realized and proved by Saito in [7]. If $x_0 \in |X|$ is at most an isolated characteristic point of f with respect to the singular support of $\mathcal{F}$, then Saito's theorem [7, Theorem 5.9] says

$$(CC(\mathcal{F}), df)_{T^*X, x_0} = -\mathrm{dimtot} R\Phi_{\bar{x}_0}(f, \mathcal{F}), \tag{1.1.2}$$

where $CC(\mathcal{F})$ is the characteristic cycle of $\mathcal{F}$. Now we propose the following question:

Question 1.2. Is there a Milnor type formula for non-isolated singular/characteristic points?

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1.3. If f is a projective flat morphism and if f is smooth outside $f^{-1}(y)$ for a closed point y of the curve Y, then the conductor formula of Bloch (cf. [8, Theorem 2.2.3 and Corollary 2.2.4])

$$-a_y(Rf_*\Lambda) = (-1)^n(X,X)_{T^*X,X_y} = (-1)^n \mathrm{deg} c^X_{n,X_y}(\Omega^1_{X/Y}) \cap [X] \tag{1.3.1}$$

gives a partial answer to the Question 1.1.2. We view (1.1.1), (1.1.2) and (1.3.1) in the form

$$\text{deg(Geometric class on singular locus)} = \text{deg(Cohomology class on singular locus)}. \tag{1.3.2}$$

In a joint work with Yigeng Zhao [12], we introduce a (cohomological) non-acyclicity class which is supported on non-acyclicity locus. Let $X \to Y$ be a separated morphism between schemes of finite type over k. Let $Z \subseteq X$ be a closed subscheme and $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $X \backslash Z \to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$. Then the cohomological non-acyclicity class $\widetilde{C}^Z_{X/Y/k}(\mathcal{F})$ is a class supported on Z (in $H^0_Z(X, \mathcal{K}_{X/Y/k})$). In a joint work with Jiangnan Xiong [10], we construct its geometric counterpart. More precisely, when f is a morphism between smooth schemes over k such that $X \to S$ is $SS(\mathcal{F})$-transversal outside Z, then we construct a class $cc^Z_{X/Y/k}(\mathcal{F}) \in \mathrm{CH}_0(Z)$ (cf. (5.5.8)), called the geometric non-acyclicity class of $\mathcal{F}$. If moreover $\dim Z < \dim Y$, then we have the following fibration formula (5.5.8)

$$cc_{X/k}(\mathcal{F}) = c_{\mathrm{dimY}}(f^*\Omega^{1,\vee}_{Y/k}) \cap cc_{X/Y}(\mathcal{F}) + cc^Z_{X/Y/k}(\mathcal{F}). \tag{1.3.3}$$

We prove that the formation of the geometric non-acyclicity class is compatible with pullback (5.9.2) and proper push-forward (5.11.1). It also satisfies Saito's Milnor formula (5.7.1) and a conductor formula (5.12.1). It is natural to expect the following conjecture holds:

Conjecture 1.4 (Conjecture 5.8)**.** *We have*

$$\widetilde{C}^Z_{X/Y/k}(\mathcal{F}) = \widetilde{\mathrm{cl}}(cc^Z_{X/Y/k}(\mathcal{F})) \quad \text{in} \quad \mathrm{CH}_0(Z), \tag{1.4.1}$$

where $\widetilde{\mathrm{cl}} : \mathrm{CH}_0(Z) \to H^0_Z(X, \mathcal{K}_{X/Y/k})$ *is the cycle class map.*

We hope (1.4.1) gives a answer to Question 1.2 in some sense.

Notation and Conventions.

(1) Let S be a Noetherian scheme and Sch_S the category of separated schemes of finite type over S. Let Λ be a Noetherian ring such that $m\Lambda = 0$ for some integer m invertible on S unless otherwise stated explicitly.
(2) For any scheme $X \in \mathrm{Sch}_S$, we denote by $D_{\mathrm{ctf}}(X, \Lambda)$ the derived category of complexes of Λ-modules of finite tor-dimension with constructible cohomology groups on X.
(3) For any separated morphism $f : X \to Y$ in Sch_S, we use the following notation

$$\mathcal{K}_{X/Y} = Rf^!\Lambda, \quad D_{X/Y}(-) = R\mathcal{H}om(-, \mathcal{K}_{X/Y}).$$

(4) To simplify our notation, we omit to write R or L to denote the derived functors unless otherwise stated explicitly or for $R\mathcal{H}om$.

2. Cohomological non-acyclicity class

2.1. Consider a commutative diagram in Sch_S:

$$\begin{array}{ccccc} Z & \xhookrightarrow{\tau} & X & \xrightarrow{f} & Y, \\ & & & \searrow^{h} \quad \swarrow_{g} & \\ & & & S & \end{array} \tag{2.1.1}$$

where $\tau : Z \to X$ is a closed immersion and g is a smooth morphism. Let us denote the diagram (2.1.1) simply by $\Delta = \Delta^Z_{X/Y/S}$ Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $X\backslash Z \to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$ and that $h : X \to S$ is universally locally acyclic relatively to $\mathcal{F}$.

2.2. In [12], we introduce an object $\mathcal{K}_\Delta = \mathcal{K}_{X/Y/S}$ sitting in a distinguished triangle (cf. [12, (4.2.5)])

$$\mathcal{K}_{X/Y} \to \mathcal{K}_{X/S} \to \mathcal{K}_\Delta \xrightarrow{+1} . \tag{2.2.1}$$

and a cohomological class $C^Z_\Delta(\mathcal{F}) = \widetilde{C}^Z_{X/Y/S}(\mathcal{F})$ in $H^0_Z(X, \mathcal{K}_\Delta)$. We call $C^Z_\Delta(\mathcal{F})$ the non-acyclicity class of $\mathcal{F}$. If the following condition holds:

$$H^0(Z, \mathcal{K}_{Z/Y}) = 0 \text{ and } H^1(Z, \mathcal{K}_{Z/Y}) = 0 \tag{2.2.2}$$

then the map $H^0_Z(X, \mathcal{K}_{X/S}) \xrightarrow{(2.2.1)} H^0_Z(X, \mathcal{K}_{X/Y/S})$ is an isomorphism. In this case, the class $\widetilde{C}^Z_{X/Y/S}(\mathcal{F}) \in H^0_Z(X, \mathcal{K}_{X/Y/S})$ defines an element of $H^0_Z(X, \mathcal{K}_{X/S})$. Now we summarize the functorial properties for the non-acyclicity classes (cf. [12, Theorem 1.9, Proposition 1.11, Theorem 1.12, Theorem 1.14]).

Proposition 2.3. *Let us denote the diagram (4.2.1) simply by $\Delta = \Delta^Z_{X/Y/S}$ and $\widetilde{C}^Z_{X/Y/S}(\mathcal{F})$ by $C_\Delta(\mathcal{F})$. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. Assume that $Y \to S$ is smooth, $X\backslash Z \to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$ and that $X \to S$ is universally locally acyclic relatively to $\mathcal{F}$.*

(1) *(Fibration formula) If $H^0(Z, \mathcal{K}_{Z/Y}) = H^1(Z, \mathcal{K}_{Z/Y}) = 0$, then we have*

$$C_{X/S}(\mathcal{F}) = c_r(f^*\Omega^{1,\vee}_{Y/S}) \cap C_{X/Y}(\mathcal{F}) + C_\Delta(\mathcal{F}) \quad \text{in} \quad H^0(X, \mathcal{K}_{X/S}). \tag{2.3.1}$$

(2) *(Pull-back) Let $b : S' \to S$ be a morphism of Noetherian schemes. Let $\Delta' = \Delta^{Z'}_{X'/Y'/S'}$ be the base change of $\Delta = \Delta^Z_{X/Y/S}$ by $b : S' \to S$. Let $b_X : X' = X \times_S S' \to X$ be the base change of b by $X \to S$. Then we have*

$$b^*_X C_\Delta(\mathcal{F}) = C_{\Delta'}(b^*_X\mathcal{F}) \quad \text{in} \quad H^0_{Z'}(X', \mathcal{K}_{X'/Y'/S'}), \tag{2.3.2}$$

where $b^*_X : H^0_Z(X, \mathcal{K}_{X/Y/S}) \to H^0_{Z'}(X', \mathcal{K}_{X'/Y'/S'})$ is the induced pull-back morphism.

(3) *(Proper push-forward) Consider a diagram $\Delta' = \Delta^{Z'}_{X'/Y/S}$. Let $s : X \to X'$ be a proper morphism over Y such that $Z \subseteq s^{-1}(Z')$. Then we have*

$$s_*(C_\Delta(\mathcal{F})) = C_{\Delta'}(Rs_*\mathcal{F}) \quad \text{in} \quad H^0_{Z'}(X', \mathcal{K}_{X'/Y/S}), \tag{2.3.3}$$

where $s_* : H^0_Z(X, \mathcal{K}_{X/Y/S}) \to H^0_{Z'}(X', \mathcal{K}_{X'/Y/S})$ is the induced push-forward morphism.

(4) *(Cohomological Milnor formula) Assume $S = \mathrm{Spec} k$ for a perfect field k of characteristic $p > 0$ and Λ is a finite local ring such that the characteristic of the residue field is invertible in k. If Y is a smooth connected curve over k and $Z = \{x\}$, then we have*

$$C_\Delta(\mathcal{F}) = -\mathrm{dimtot} R\Phi_{\bar{x}}(\mathcal{F}, f) \quad \text{in} \quad \Lambda = H^0_x(X, \mathcal{K}_{X/k}), \tag{2.3.4}$$

where $R\Phi(\mathcal{F}, f)$ is the complex of vanishing cycles and $\mathrm{dimtot} = \dim + \mathrm{Sw}$ is the total dimension.

(5) *(Cohomological conductor formula) Assume $S = \mathrm{Spec} k$ for a perfect field k of characteristic $p > 0$ and Λ is a finite local ring such that the characteristic of the residue field is invertible in k. If Y is a smooth connected curve over k and $Z = f^{-1}(y)$ for a closed point $y \in |Y|$, then we have*

$$f_* C_\Delta(\mathcal{F}) = -a_y(Rf_*\mathcal{F}) \quad \text{in} \quad \Lambda = H^0_y(Y, \mathcal{K}_{Y/k}), \tag{2.3.5}$$

where $a_y(\mathcal{G}) = \mathrm{rank}\mathcal{G}|_{\bar{\eta}} - \mathrm{rank}\mathcal{G}_{\bar{y}} + \mathrm{Sw}_y\mathcal{G}$ *is the Artin conductor of the object* $\mathcal{G} \in D_{\mathrm{ctf}}(Y, \Lambda)$ *at* y *and* η *is the generic point of* Y*.*

The formation of non-acyclicity classes is also compatible with specialization maps (cf. [12, Proposition 4.17]). We call (2.3.1) the fibration formula for characteristic class, which is motivated from [9].

2.4. Let X be a smooth connected curve over k. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ and $Z \subseteq X$ be a finite set of closed points such that the cohomology sheaves of $\mathcal{F}|_{X\setminus Z}$ are locally constant. By the cohomological Milnor formula (2.3.4), we have the following (motivic) expression for the Artin conductor of $\mathcal{F}$ at $x \in Z$

$$a_x(\mathcal{F}) = \mathrm{dimtot} R\Phi_{\bar{x}}(\mathcal{F}, \mathrm{id}) = -C^{\{x\}}_{U/U/k}(\mathcal{F}|_U), \tag{2.4.1}$$

where U is any open subscheme of X such that $U \cap Z = \{x\}$. By (2.3.1), we get the following cohomological Grothendieck-Ogg-Shafarevich formula (cf. [12, Corollary 6.6]):

$$C_{X/k}(\mathcal{F}) = \mathrm{rank}\mathcal{F} \cdot c_1(\Omega^{1,\vee}_{X/k}) - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \quad \text{in} \quad H^0(X, \mathcal{K}_{X/k}). \tag{2.4.2}$$

3. Transversality condition

3.1. We recall the transversality condition introduced in [12, 2.1], which is a relative version of the transversality condition studied by Saito [7, Definition 8.5]. Consider the following cartesian diagram in Sch_S:

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle f} \\ W & \xrightarrow{\delta} & T. \end{array} \tag{3.1.1}$$

Let $\mathcal{F} \in D_{\mathrm{ctf}}(Y, \Lambda)$ and $\mathcal{G} \in D_{\mathrm{ctf}}(T, \Lambda)$. Let $c_{\delta,f,\mathcal{F},\mathcal{G}}$ be the composition

$$\begin{aligned} c_{\delta,f,\mathcal{F},\mathcal{G}} : i^*\mathcal{F} \otimes^L p^*\delta^!\mathcal{G} &\xrightarrow{id\otimes \mathrm{b.c}} i^*\mathcal{F} \otimes^L i^! f^*\mathcal{G} \\ &\xrightarrow{\mathrm{adj}} i^! i_!(i^*\mathcal{F} \otimes^L i^! f^*\mathcal{G}) \\ &\xrightarrow[\simeq]{\mathrm{proj.formula}} i^!(\mathcal{F} \otimes^L i_! i^! f^*\mathcal{G}) \xrightarrow{\mathrm{adj}} i^!(\mathcal{F} \otimes^L f^*\mathcal{G}). \end{aligned} \tag{3.1.2}$$

We put $c_{\delta,f,\mathcal{F}} := c_{\delta,f,\mathcal{F},\Lambda} : i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \to i^!\mathcal{F}$. If $c_{\delta,f,\mathcal{F}}$ is an isomorphism, then we say that the morphism δ is $\mathcal{F}$*-transversal*.

By [12, 2.11], there is a functor $\delta^\Delta : D_{\mathrm{ctf}}(Y, \Lambda) \to D_{\mathrm{ctf}}(X, \Lambda)$ such that for any $\mathcal{F} \in D_{\mathrm{ctf}}(Y, \Lambda)$, we have a distinguished triangle

$$i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \xrightarrow{c_{\delta,f,\mathcal{F}}} i^!\mathcal{F} \to \delta^\Delta\mathcal{F} \xrightarrow{+1} . \tag{3.1.3}$$

δ is $\mathcal{F}$-transversal if and only if $\delta^\Delta(\mathcal{F})$=0 (cf. [12, Lemma 2.12]).

The following lemma gives an equivalence between transversality condition and (universally) locally acyclicity condition.

Lemma 3.2. *Let* $f : X \to S$ *be a morphism of finite type between Noetherian schemes and* $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$*. The following conditions are equivalent:*

(1) *The morphism* f *is locally acyclic relatively to* $\mathcal{F}$*.*
(2) *The morphism* f *is universally locally acyclic relatively to* $\mathcal{F}$*.*

(3) *For any* $\mathcal{G} \in D_{\mathrm{ctf}}(X, \Lambda)$*, the canonical map*

$$D_{X/S}(\mathcal{G}) \boxtimes^L \mathcal{F} \to R\mathcal{H}om(\mathrm{pr}_1^* \mathcal{G}, \mathrm{pr}_2^! \mathcal{F}) \tag{3.2.1}$$

is an isomorphism.

(4) *The canonical map*

$$D_{X/S}(\mathcal{F}) \boxtimes^L \mathcal{F} \to R\mathcal{H}om(\mathrm{pr}_1^* \mathcal{F}, \mathrm{pr}_2^! \mathcal{F}) \tag{3.2.2}$$

is an isomorphism.

(5) *For any cartesian diagram between Noetherian schemes*

$$\begin{array}{ccc} Y \times_S X & \xrightarrow{\mathrm{pr}_2} & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \Box & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\delta]{} & S \end{array} \tag{3.2.3}$$

the morphism δ *is* $\mathcal{F}$*-transversal.*

(6) *For any cartesian diagram (3.2.3) and any* $\mathcal{G} \in D_{\mathrm{ctf}}(S, \Lambda)$*, the morphism* $c_{\delta,f,\mathcal{F},\mathcal{G}}$ *is an isomorphism.*

(7) *For any cartesian diagram between Noetherian schemes*

$$\begin{array}{ccccc} Y \times_S X & \xrightarrow{\mathrm{pr}_2} & X' & \longrightarrow & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \Box & \downarrow{\scriptstyle f'} & \Box & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\delta]{} & S' & \longrightarrow & S, \end{array} \tag{3.2.4}$$

the morphism δ *is* $\mathcal{F}|_{X'}$*-transversal.*

(8) *For any cartesian diagram (3.2.4) and any* $\mathcal{G} \in D_{\mathrm{ctf}}(S, \Lambda)$*, the morphism* $c_{\delta,f,\mathcal{F},\mathcal{G}}$ *is an isomorphism.*

When S is a scheme of finite type over a field k, then the equivalence between (2) and (7) follows from [12, Proposition 2.4.(2) and Proposition 2.5]. In this case, we may require Y and S' smooth over k in (7).

4. Non-acyclicity classes

4.1. Let S be a Noetherian scheme and Sch_S the category of separated schemes of finite type over S. Let Λ be a Noetherian ring such that $m\Lambda = 0$ for some integer m invertible on S. Consider the following cartesian diagram in Sch_S

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\mathrm{pr}_1} & X \\ {\scriptstyle \mathrm{pr}_2}\downarrow & \Box & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & S, \end{array} \tag{4.1.1}$$

where pr_1 and pr_2 are the projections. For any $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ and $\mathcal{G} \in D_{\mathrm{ctf}}(Y, \Lambda)$, we have canonical morphisms

$$\mathcal{F} \boxtimes_S^L \mathcal{K}_{Y/S} = \mathrm{pr}_1^* \mathcal{F} \otimes^L \mathrm{pr}_2^* g^! \Lambda \xrightarrow{c_{g,h,\mathcal{F}}} \mathrm{pr}_1^! \mathcal{F}, \tag{4.1.2}$$

$$\mathcal{F} \boxtimes_S^L D_{Y/S}(\mathcal{G}) \to R\mathcal{H}om(\mathrm{pr}_2^* \mathcal{G}, \mathrm{pr}_1^! \mathcal{F}), \tag{4.1.3}$$

where (4.1.3) is adjoint to

$$\mathcal{F}\boxtimes_S^L (D_{Y/S}(\mathcal{G})\otimes^L \mathcal{G}) \xrightarrow{id\boxtimes \mathrm{ev}} \mathcal{F}\boxtimes_S^L \mathcal{K}_{Y/S} \xrightarrow{(4.1.2)} \mathrm{pr}_1^!\mathcal{F}. \tag{4.1.4}$$

Note that (4.1.2) is a special case of (4.1.3) by taking $\mathcal{G}=\Lambda$. If moreover $X\to S$ is universally locally acyclic relatively to $\mathcal{F}$, then (4.1.3) is an isomorphism by [6, Proposition 2.5](see also [11, Corollary 3.1.5]). For a morphism $c=(c_1,c_2):C\to X\times_S Y)$, we have a canonical isomorphism by [3, Corollaire 3.1.12.2]

$$R\mathcal{H}om(c_2^*\mathcal{G}, c_1^!\mathcal{F}) \xrightarrow{\simeq} c^! R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{G}, \mathrm{pr}_1^!\mathcal{F}). \tag{4.1.5}$$

4.2. Consider a commutative diagram in Sch_S:

$$\begin{array}{ccccc} Z & \overset{\tau}{\hookrightarrow} & X & \xrightarrow{f} & Y, \\ & & & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & & & S & \end{array} \tag{4.2.1}$$

where $\tau: Z\to X$ is a closed immersion and g is a smooth morphism. Let $i: X\times_Y X\to X\times_S X$ be the base change of the diagonal morphism $\delta: Y\to Y\times_S Y$:

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle \delta_1}\downarrow & \square & \downarrow{\scriptstyle \delta_0} \\ X\times_Y X & \xrightarrow{i} & X\times_S X \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle f\times f} \\ Y & \xrightarrow{\delta} & Y\times_S Y, \end{array} \qquad (f: X\to Y) \tag{4.2.2}$$

where δ_0 and δ_1 are the diagonal morphisms. Put $\mathcal{K}_{X/Y/S}:=\delta^\Delta\mathcal{K}_{X/S}\simeq \delta_1^*\delta^\Delta\delta_{0*}\mathcal{K}_{X/S}$. We have the following distinguished triangle (cf. [12, (4.2.5)])

$$\mathcal{K}_{X/Y}\to\mathcal{K}_{X/S}\to\mathcal{K}_{X/Y/S}\xrightarrow{+1}. \tag{4.2.3}$$

Let $\mathcal{F}\in D_{\mathrm{ctf}}(X,\Lambda)$ such that $X\backslash Z\to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$ and that $h: X\to S$ is universally locally acyclic relatively to $\mathcal{F}$. We put

$$\mathcal{H}_S = R\mathcal{H}om_{X\times_S X}(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^!\mathcal{F}), \qquad \mathcal{T}_S=\mathcal{F}\boxtimes_S^L D_{X/S}(\mathcal{F}). \tag{4.2.4}$$

Lemma 4.3. *$\delta_1^*\delta^\Delta\mathcal{T}_S$ is supported on Z.*

The relative cohomological characteristic class $C_{X/S}(\mathcal{F})$ is the composition (cf. [12, 3.1])

$$\Lambda\xrightarrow{\mathrm{id}} R\mathcal{H}om(\mathcal{F},\mathcal{F}) \underset{\simeq}{\xrightarrow{(4.1.5)}} \delta_0^!\mathcal{H}_S \underset{\simeq}{\xleftarrow{(4.1.3)}} \delta_0^!\mathcal{T}_S\to\delta_0^*\mathcal{T}_S\xrightarrow{\mathrm{ev}}\mathcal{K}_{X/S}. \tag{4.3.1}$$

The non-acyclicity class $\widetilde{C}^Z_{X/Y/S}(\mathcal{F})$ is the composition (cf. [12, Definition 4.6])

$$\Lambda\to\delta_0^!\mathcal{H}_S\xleftarrow{\simeq}\delta_0^!\mathcal{T}_S\simeq\delta_1^! i^!\mathcal{T}_S\to\delta_1^* i^!\mathcal{T}_S\to\delta_1^*\delta^\Delta\mathcal{T}_S\xleftarrow{\simeq}\tau_*\tau^!\delta_1^*\delta^\Delta\mathcal{T}_S\to\tau_*\tau^!\mathcal{K}_{X/Y/S}. \tag{4.3.2}$$

5. Geometric non-acyclicity class

Now we construct a geometric counterpart of the cohomological non-acyclicity class. Let k be a perfect field of characteristic p and Λ be a finite local ring whose residue field is of characteristic $\ell\neq p$. We first recall geometric transversal condition.

5.1. Let X be a smooth scheme of dimension d over k and $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. We need Beilinson's singular support $SS(\mathcal{F})$, which a d-dimensional conical closed subset of the cotangent bundle T^*X). We also need Saito's characteristic cycle $CC(\mathcal{F})$, which is a d-cycle supported on $SS(\mathcal{F})$ with integral coefficients. The characteristic cycle $CC(\mathcal{F})$ is characterized by a Milnor formula for isolated characteristic points.

We say a morphism $f : X \to S$ to a smooth scheme S is $SS(\mathcal{F})$-transversal if $df^{-1}(SS(\mathcal{F}))$ is contained in the zero section of $T^*S \times_S X$, where $df : T^*S \times_S X \to T^*X$ is induced morphism on vector bundles. We have the following fact:

Lemma 5.2. *If $f : X \to S$ is $SS(\mathcal{F})$-transversal, then f is universally locally acyclic relatively to $\mathcal{F}$.*

5.3. Let S be a smooth connected scheme of dimension s over k. Let $f : X \to S$ be a morphism in Sm_k. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that f is $SS(\mathcal{F})$-transversal. Consider the following morphisms

$$X \xrightarrow{0} T^*S \times_S X \xrightarrow{df} T^*X, \tag{5.3.1}$$

where 0 stands for the zero section. By assumption $df^{-1}(SS(\mathcal{F}))$ is contained in $0(X)$. We define the relative characteristic class of $\mathcal{F}$ to be the following s-cycle class on X:

$$cc_{X/S}(\mathcal{F}) := (-1)^s \cdot (df)^!(CC(\mathcal{F})) \quad \text{in} \quad \mathrm{CH}_s(X), \tag{5.3.2}$$

where $(df)^!$ is the refined Gysin pullback. We don't know how to define $cc_{X/S}(\mathcal{F})$ if one only assume f is universally locally acyclic relatively to $\mathcal{F}$.

If f is a smooth morphism of relative dimension r and if $\mathcal{F}$ is locally constant, then we have

$$cc_{X/S}(\mathcal{F}) = (-1)^s \cdot 0^!_{X/S}((-1)^{\dim X} \cdot \mathrm{rank}\mathcal{F} \cdot [X]) = \mathrm{rank}\mathcal{F} \cdot c_r(\Omega^{1,\vee}_{X/S}) \cap [X]. \tag{5.3.3}$$

We propose the following conjecture:

Conjecture 5.4. *Let S be a smooth connected scheme of dimension s over k. Let $f : X \to S$ be a morphism in Sm_k. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that f is $SS(\mathcal{F})$-transversal. Then we have*

$$\mathrm{cl}(cc_{X/S}(\mathcal{F})) = C_{X/S}(\mathcal{F}) \quad \text{in} \quad H^0(X, \mathcal{K}_{X/S}), \tag{5.4.1}$$

where $\mathrm{cl} : \mathrm{CH}_s(X) \to H^0(X, \mathcal{K}_{X/S})$ is the cycle class map.

When $S = \mathrm{Spec}k$, then it is Saito's conjecture [7, Conjecture 6.8.1], which is proved under quasi-projective assumption in [12, Theorem 1.3]. When $f : X \to S$ is a smooth morphism, then (5.4.1) is true for a locally constant constructible (flat) sheaf $\mathcal{F}$ of Λ-modules. Indeed, this follows from (5.3.3), [12, Lemma 3.3] and (2.3.1).

5.5. Consider a commutative diagram in Sm_k:

$$\begin{array}{ccccc} Z & \xhookrightarrow{\tau} & X & \xrightarrow{f} & Y \\ & & & \searrow^{h} \quad \swarrow_{g} & \\ & & & S & \end{array}, \tag{5.5.1}$$

where $\tau : Z \to X$ is a closed immersion and g is a smooth morphism of relative dimension r. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $X \backslash Z \to Y$ is $SS(\mathcal{F}|_{X \backslash Z})$-transversal and that $X \to S$ is $SS(\mathcal{F})$-transversal.

We have a commutative diagram on vector bundles

$$
\begin{array}{ccccc}
X & = = = & X & & \\
\downarrow & \square & \downarrow{\scriptstyle 0} & & \\
T^*S\times_S X & \xrightarrow{dg_X} & T^*Y\times_Y X & \xrightarrow{df} & T^*X \\
\downarrow & \square & \downarrow & & \\
T^*S\times_S Y & \xrightarrow{dg} & T^*Y & & \\
\downarrow & \square & \downarrow & & \\
Y & \xrightarrow{0} & T^*(Y/S), & &
\end{array}
\tag{5.5.2}
$$

where dg_X is the base change of dg. By assumption, $df^{-1}(SS(\mathcal{F}))$ is supported on $0(X)\cup T^*Y\times_Y Z$ and $dh^{-1}(SS(\mathcal{F}))={dg_X}^{-1}df^{-1}(SS(\mathcal{F}))$ is contained in the zero section $0(X)\subseteq T^*S\times_S X$. We define the geometric non-acyclicity class $cc^Z_{X/Y/S}(\mathcal{F})$ of $\mathcal{F}$ to be

$$cc^Z_{X/Y/S}(\mathcal{F}):=(-1)^s\cdot dg_X^!(df^!(CC(\mathcal{F}))|_{T^*Y\times_Y Z})\quad\text{in}\quad \mathrm{CH}_s(Z). \tag{5.5.3}$$

Assume moreover that $\dim Z<r+s$. Then the restriction map $\mathrm{CH}_{r+s}(X)\xrightarrow{\simeq}\mathrm{CH}_{r+s}(X\backslash Z)$ is an isomorphism. In this case, we define the relative characteristic class $cc_{X/Y}(\mathcal{F})$ to be

$$cc_{X/Y}(\mathcal{F}):=cc_{U/Y}(\mathcal{F}|_U)\quad\text{in}\quad \mathrm{CH}_{r+s}(X), \tag{5.5.4}$$

where $U=X\backslash Z$. Then we have

$$(-1)^s\cdot df^!(CC(\mathcal{F}))=cc_{X/Y}(\mathcal{F})+(-1)^s\cdot df^!(CC(\mathcal{F}))|_{T^*Y\times_Y Z}, \tag{5.5.5}$$

$$cc_{X/S}(\mathcal{F})=(-1)^s\cdot dg_X^! df^!(CC(\mathcal{F}))=dg_X^! cc_{X/Y}(\mathcal{F})+(-1)^s\cdot dg_X^!(df^!(CC(\mathcal{F}))|_{T^*Y\times_Y Z}), \tag{5.5.6}$$

By the excess intersection formula, we have

$$dg_X^! cc_{X/Y}(\mathcal{F})=c_r(f^*\Omega^{1,\vee}_{Y/S})\cap cc_{X/Y}(\mathcal{F}). \tag{5.5.7}$$

Thus if $\dim Z<r+s$, then we have

$$cc_{X/S}(\mathcal{F})=c_r(f^*\Omega^{1,\vee}_{Y/S})\cap cc_{X/Y}(\mathcal{F})+cc^Z_{X/Y/S}(\mathcal{F}). \tag{5.5.8}$$

In particular, if Z is empty, then we have

$$cc_{X/S}(\mathcal{F})=c_r(f^*\Omega^{1,\vee}_{Y/S})\cap cc_{X/Y}(\mathcal{F}). \tag{5.5.9}$$

Remark 5.6. Assume that $X\to S$ is smooth of relative dimension r and that $X\backslash Z\to Y$ is smooth of relative dimension n ($n<r$). Then $\Omega^{1,\vee}_{X/Y}$ is locally free of rank n on $X\backslash Z$ and we have the localized Chern classes $c^X_{i,Z}(\Omega^{1,\vee}_{X/Y})$ for $i>n$ (cf. [2, Section 1]). By [8, Lemma 2.1.4], we have

$$cc^Z_{X/Y/S}(\Lambda)=(-1)^r c^X_{r,Z}(\Omega^1_{X/Y})\cap[X]\quad\text{in}\quad \mathrm{CH}_s(Z). \tag{5.6.1}$$

Theorem 5.7 (Saito's Milnor formula)**.** *Assume $S=\mathrm{Spec}k$, $Y=\mathbb{A}^1_k$ and $Z=\{x\}$. Then we have*

$$cc^Z_{X/Y/S}(\mathcal{F})=-\mathrm{dimtot}R\Phi_{\bar{x}}(\mathcal{F},f)\quad\text{in}\quad \mathbb{Z}=\mathrm{CH}_0(\{x\}). \tag{5.7.1}$$

We expect the following Milnor type formula for non-isolated singular/characteristic points holds.

Conjecture 5.8. *Let S be a smooth connected k-scheme of dimension s. Consider the commutative diagram (5.5.1). Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $X\backslash Z \to Y$ is $SS(\mathcal{F}|_{X\backslash Z})$-transversal and that $X \to S$ is $SS(\mathcal{F})$-transversal. Then we have an equality*

$$\widetilde{C}^Z_{X/Y/S}(\mathcal{F}) = \widetilde{\mathrm{cl}}(cc^Z_{X/Y/S}(\mathcal{F})) \quad \text{in} \quad H^0_Z(X, \mathcal{K}_{X/Y/S}), \tag{5.8.1}$$

where $\widetilde{\mathrm{cl}}$ is the composition $CH_s(Z) \xrightarrow{\mathrm{cl}} H^0_Z(X, \mathcal{K}_{X/S}) \xrightarrow{(4.2.3)} H^0_Z(X, \mathcal{K}_{X/Y/S})$.

When $S = \mathrm{Spec}k$, $Y = \mathbb{A}^1_k$ and $Z = \{x\}$, then Conjecture 5.8 follows from Saito's Milnor formula (5.7.1) and the cohomological Milnor formula (2.3.4).

Proposition 5.9. *Consider a commutative diagram in* Sm_k

$$\begin{array}{ccccc} X' & & \xrightarrow{i_X} & & X \\ & \searrow^{f'} & & & & \searrow^{f} \\ \downarrow^{h'} & & Y' & \xrightarrow{i_Y} & \downarrow^{h} & & Y \\ & \swarrow_{g'} & & & & \swarrow_{g} \\ S' & & \xrightarrow{\delta} & & S, \end{array} \tag{5.9.1}$$

where squares are cartesian diagrams. Let $Z \subseteq X$ be a closed subscheme and $Z' = Z \times_X X'$. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $X \to S$ is $SS(\mathcal{F})$-transversal and $X\backslash Z \to Y$ is $SS(\mathcal{F}|_{X\backslash Z})$-transversal. Assume that f and g are smooth morphisms and that i_X is properly $SS(\mathcal{F})$-transversal. Assume S (resp. S') is connected of dimension s (resp. s'). Then we have

$$i^!_X cc^Z_{X/Y/S}(\mathcal{F}) = cc^{Z'}_{X'/Y'/S'}(i^*_X\mathcal{F}) \quad \text{in} \quad CH_{s'}(Z'), \tag{5.9.2}$$

where $i^!_X \colon CH_s(Z) \to CH_{s'}(Z')$ is the refined Gysin pull-back.

5.10. Let $g : Y \to S$ be a smooth morphism in Sm_k. Consider a commutative diagram in Sm_k:

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ & {}_{f}\searrow \quad \swarrow_{f'} & \\ & Y & \end{array}. \tag{5.10.1}$$

Let $Z \subseteq X$ be a closed subscheme. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $X \to S$ is $SS(\mathcal{F})$-transversal and that $X\backslash Z \to Y$ is $SS(\mathcal{F}|_Z)$-transversal. Assume p is a proper morphism and put $Z' = p(Z)$. By [7, Lemma 3.8 and Lemma 4.2.6], the morphism $X' \to S$ is $SS(Rp_*\mathcal{F})$-transversal and that $X'\backslash Z' \to Y$ is $SS(Rp_*\mathcal{F}|_Z)$-transversal. Then we have well defined classes $cc^Z_{X/Y/S}(\mathcal{F}) \in \mathrm{CH}_s(Z)$ and $cc^{Z'}_{X'/Y/S}(Rp_*\mathcal{F}) \in \mathrm{CH}_s(Z')$.

Proposition 5.11. *Consider the assumptions in 5.10. Assume moreover $\dim p_\circ SS(\mathcal{F}) \leqslant \dim X'$, Y is projective and p is quasi-projective. Then we have*

$$p_* cc^Z_{X/Y/S}(\mathcal{F}) = cc^{Z'}_{X'/Y/S}(Rp_*\mathcal{F}), \tag{5.11.1}$$

where $p_* : \mathrm{CH}_s(Z) \to \mathrm{CH}_s(Z')$ is the proper push-forward.

Corollary 5.12 (Saito, [8, Theorem 2.2.3])**.** *Let $f : X \to Y$ be a projective morphism of smooth schemes over a perfect field k, and let $y \in Y$ be a closed point. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. Assume Y is a smooth and connected curve and that f is properly $SS(\mathcal{F})$-transversal outside X_y. Then we have*

$$-a_y(Rf_*\mathcal{F}) = f_* cc^{X_y}_{X/Y/k}(\mathcal{F}). \tag{5.12.1}$$

References

[1] A. Beilinson, *Constructible sheaves are holonomic*, Sel. Math. New Ser. 22, (2016): 1797–1819.

[2] S. Bloch, *Cycles on arithmetic schemes and Euler characteristics of curves*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, (1987): 421–450. ↑8

[3] P. Deligne, *La formule de dualité globale*, Exposé XVIII, pp.481-587 in SGA4 Tome 3: *Théorie des topos et cohomologie étale des schémas*, edited by M.Artin et al., Lecture Notes in Math.305, Springer, 1973. ↑6

[4] P. Deligne, *La formule de Milnor*, Exposé XVI, pp.197-211 in SGA7 II: *Groupes de Monodromie en Géométrie Algébrique*, Lecture Notes in Math. 340, Springer, 1973. ↑1

[5] P. Deligne, *Notes sur Euler-Poincaré: brouillon project*, 8/2/2011. ↑1

[6] Q. Lu, W. Zheng, *Categorical traces and a relative Lefschetz-Verdier formula*, Forum of Mathematics, Sigma, Vol.10 (2022): 1-24. ↑6

[7] T. Saito, *The characteristic cycle and the singular support of a constructible sheaf*, Inventiones mathematicae, 207 (2017): 597-695. ↑1, ↑4, ↑7, ↑9

[8] T. Saito, *Characteristic cycles and the conductor of direct image*, J. Amer. Math. Soc. 34 (2021): 369-410. ↑2, ↑8, ↑9

[9] N. Umezaki, E. Yang and Y. Zhao, *Characteristic class and the ε-factor of an étale sheaf*, Trans. Amer. Math. Soc. 373 (2020): 6887-6927. ↑4

[10] J. Xiong and E. Yang, *Milnor formula for non-isolated singularities*, preprint, 2023. ↑2

[11] E. Yang and Y. Zhao, *On the relative twist formula of ℓ-adic sheaves*, Acta. Math. Sin.-English Ser. 37 (2021): 73–94. ↑6

[12] E. Yang and Y. Zhao, *Cohomological Milnor formula and Saito's conjecture on characteristic classes*, 2022, arXiv:2209.11086. ↑2, ↑3, ↑4, ↑5, ↑6, ↑7