Muskat问题的极值原理

Maximum principle for Muskat problem

报告人: 刘浩然
指导教师: 童嘉骏老师

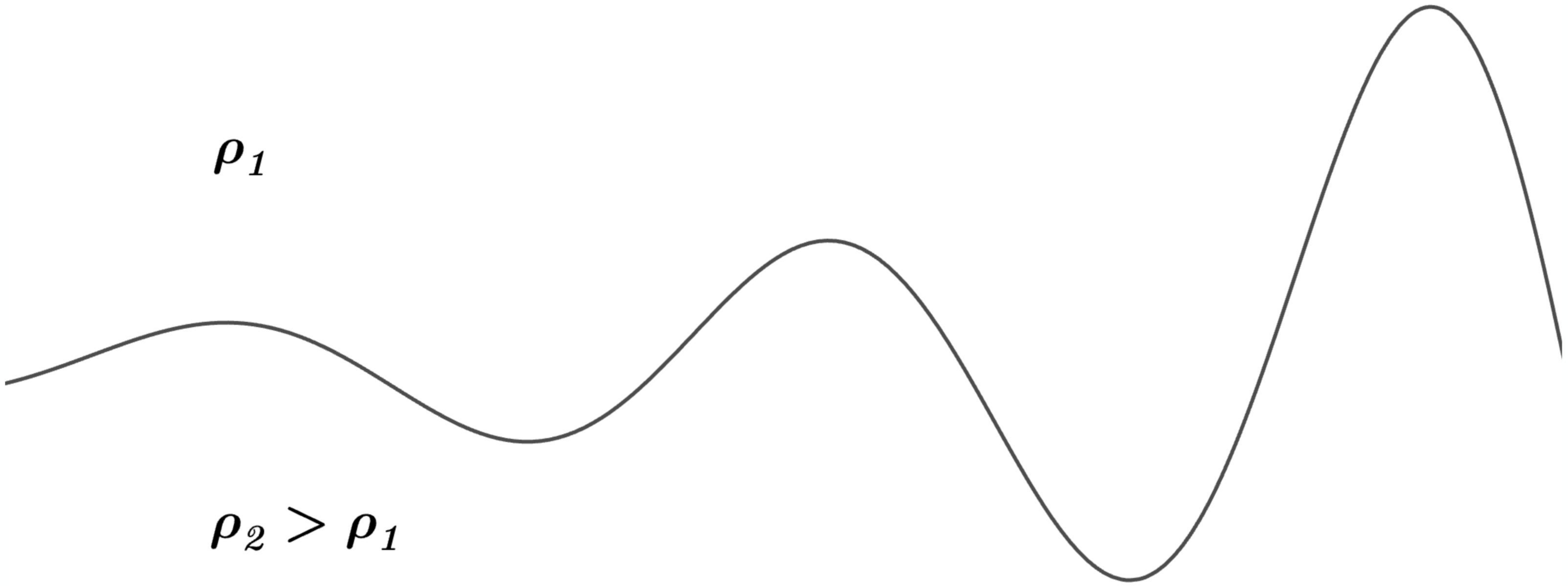
Introduction to Muskat Problem

Muskat问题

- free boundary problem
- the motion of the interface between 2 incompressible, immiscible fluids of different constant densities in a porous media

Global regularity vs finite-time singularity:

- Flatten over time?
- Finite-time turning of the interface?



Muskat问题

- the equation for the interface:

$$\partial_t f(t,x) = \int_{\mathbb{R}} \frac{(f(t,y)-f(t,x)) - (y-x)f_x(t,x)}{(f(t,y)-f(t,x))^2 + (y-x)^2} \,\mathrm{d}y$$

- linearized equation:

$$\partial_t f(t,x) = \int_{\mathbb{R}} \frac{f(t,y)-f(t,x)}{(y-x)^2} \,\mathrm{d}y$$

$$\partial_t f(t,x) = -C(-\Delta)^{\frac{1}{2}} f(t,x)$$

- Parabolic equation.

Muskat问题的已有研究成果

- [Córdoba and Gancedo 2009]: L^∞ maximum principle:

$$\| f(t,\cdot) \|_{L^\infty} \le \| f(0,\cdot) \|_{L^\infty}.$$

- [Córdoba and Gancedo 2009]: Local well-posedness in H^k for $k \ge 3$.

- [Constantin et al. 2013]: L^2 maximum principle:

$$\| f(t,\cdot) \|_{L^2} \le \| f(0,\cdot) \|_{L^2}.$$

Muskat问题的尺度变换性质

- Scaling property: If $f(t,x)$ is a solution with initial data $f_0(x)$,

$$f_\lambda(t,x) := \lambda f(\frac{t}{\lambda},\frac{x}{\lambda})$$

is a solution with initial data $f_{\lambda,0}(x) = \lambda f_0(\frac{x}{\lambda})$.

- Scaling invariant norm, or critical norm: Norm $\|\cdot\|_X$ such that

$$\left\| f_0 \right\|_X = \left\| f_{\lambda,0} \right\|_X, \quad \forall f_0 .$$

- Examples: $\|f\|_{\dot{W}^{1,\infty}} := \|f_x\|_{L^\infty}$ and $\|f\|_1 := \|\xi \hat{f}(\xi)\|_{L^1}$.

Muskat问题的已有研究成果

Well-posedness theory for scaling invariant norms:

- [Constantin et al. 2013]: Exists global weak Lipschitz solution when $\|f_0'\|_{L^\infty} < 1$.
- [Constantin et al. 2016]: Globally well-posed when $\|f_0\|_1 = \| |\xi| \hat{f}_0(\xi) \|_{L^1_\xi} \lesssim \frac{1}{3}$.
- [Constantin et al. 2017]: H^k inital data stays H^k, provided that $f_x(t, \cdot\,)$ is bounded and has some modulus of continuity.
- [Cameron 2019]: Maximum principle and global well-posedness when $\|f_0'\|_{L^\infty} < 1$.

Muskat问题的已有研究成果

What happens when $\|f_0'\|_{L^\infty} > 1$?

- Loss of parabolicity $\Longrightarrow$ finite time blow-up(?)

My results: A priori estimates of scaling invariant norm $\|f_x(t,\cdot)\|_{L^\infty}$ when $\|f_0'\|_{L^\infty} > 1$.

- Maximum principle of $\|f_x(t,\cdot)\|_{L^\infty}$ when $\|f_0'\|_{L^\infty} \leq L \approx 2.98$,
- Optimality of the constant L,
- Power law decay of $\|f_x(t,\cdot)\|_{L^\infty}$ when $\|f_0'\|_{L^\infty} < L$.

Muskat问题的极值原理

Theorem I Suppose $f(t,x) \in C_t^1([0,T], C_x^1(\mathbb{R}))$ is a smooth solution with initial data $f_0(x)$ such that $\|f_0'\|_{L_x^\infty} \leq L$, where $L \approx 2.98$ is the first positive root of the following trigonometric equation

$$\begin{cases} 2\cos(\alpha) + 3\cos(\frac{5\alpha}{3}) - \cos(3\alpha) = 0, \\ L = \tan(\alpha)\,. \end{cases}$$

Then, $f_x(t,x)$ has maximum principle. To wit, $\|f_x(t,\cdot)\|_{L_x^\infty}$ decreases in time.

Remark on theorem I: The constant $L \approx 2.98$ is the optimal constant for maximum principle to be hold. To wit, for any $\varepsilon > 0$, there exists a solution $f_\varepsilon(t,x)$, such that $\|f_{0,\varepsilon}'(0,\cdot)\|_{L_x^\infty} = L + \varepsilon$, and $\frac{d}{dt}\|f_\varepsilon(t,\cdot)\|_{L_x^\infty}\Big|_{t=0} > 0$.

Muskat问题的极值原理

Theorem II Suppose $f(t,x) \in C_t^1([0,T], C_x^1(\mathbb{R}))$ is a smooth solution with initial data $f_0(x)$ such that $\|f_0'\|_{L_x^\infty} < L$, where $L \approx 2.98$ is defined as in theorem I. Then, $\|f_x(t,\cdot)\|_{L_x^\infty}$ decays at a power law rate in time. To be specific,

$$\begin{cases} \left\| f_x(t,\cdot) \right\|_{L_x^\infty} \le \dfrac{C \left\| f_0 \right\|_{L^\infty}}{t}, \\ \left\| f_x(t,\cdot) \right\|_{L_x^\infty} \le \dfrac{C \left\| f_0 \right\|_{L^2}}{t^{3/2}}, \end{cases}$$

where C is a constant that depends only on $\|f_0'\|_{L^\infty}$.

Remark on theorem II: The power law decay rate is optimal.

Muskat问题的极值原理

- Main difficulty: the loss of parabolicity when $\|f_0'\|_{L^\infty} > 1$.
- the equation of $f_x(t,x)$:

$$\partial_t f_x(t,x) = -\partial_x f_x(t,x) \cdot \int_{\mathbb{R}} \frac{y-x}{(f(t,y)-f(t,x))^2 + (y-x)^2}\,\mathrm{d}y$$
$$-2\int_{\mathbb{R}} \frac{f_x(t,x) - l(t,y,x)}{(y-x)^2} \cdot \frac{1 + f_x(t,x)l(t,y,x)}{\left(1 + l^2(t,y,x)\right)^2}\,\mathrm{d}y,$$

where $l(t,y,x) := \frac{f(t,y)-f(t,x)}{y-x}$ is the slope between $(x, f(t,x))$ and $(y, f(t,y))$.

- If $\|f_0'\|_{L^\infty} < 1$, $1 + f_x(t,x)l(t,y,x)$ is positive $\Longrightarrow \partial_t f_x(t,x) \approx$ drift term $-\, C(-\Delta)^{\frac{1}{2}} f_x(t,x)$.
- If $\|f_0'\|_{L^\infty} > 1$, $1 + f_x(t,x)l(t,y,x)$ is indefinite $\Longrightarrow$ loss of parabolicity.

Motivation

证明的动机

- Assuming $f_x(t,x^*) = \max_{x\in\mathbb{R}} |f_x(t,x)|$, we want to show $\partial_t f_x(t,x^*) \le 0$.

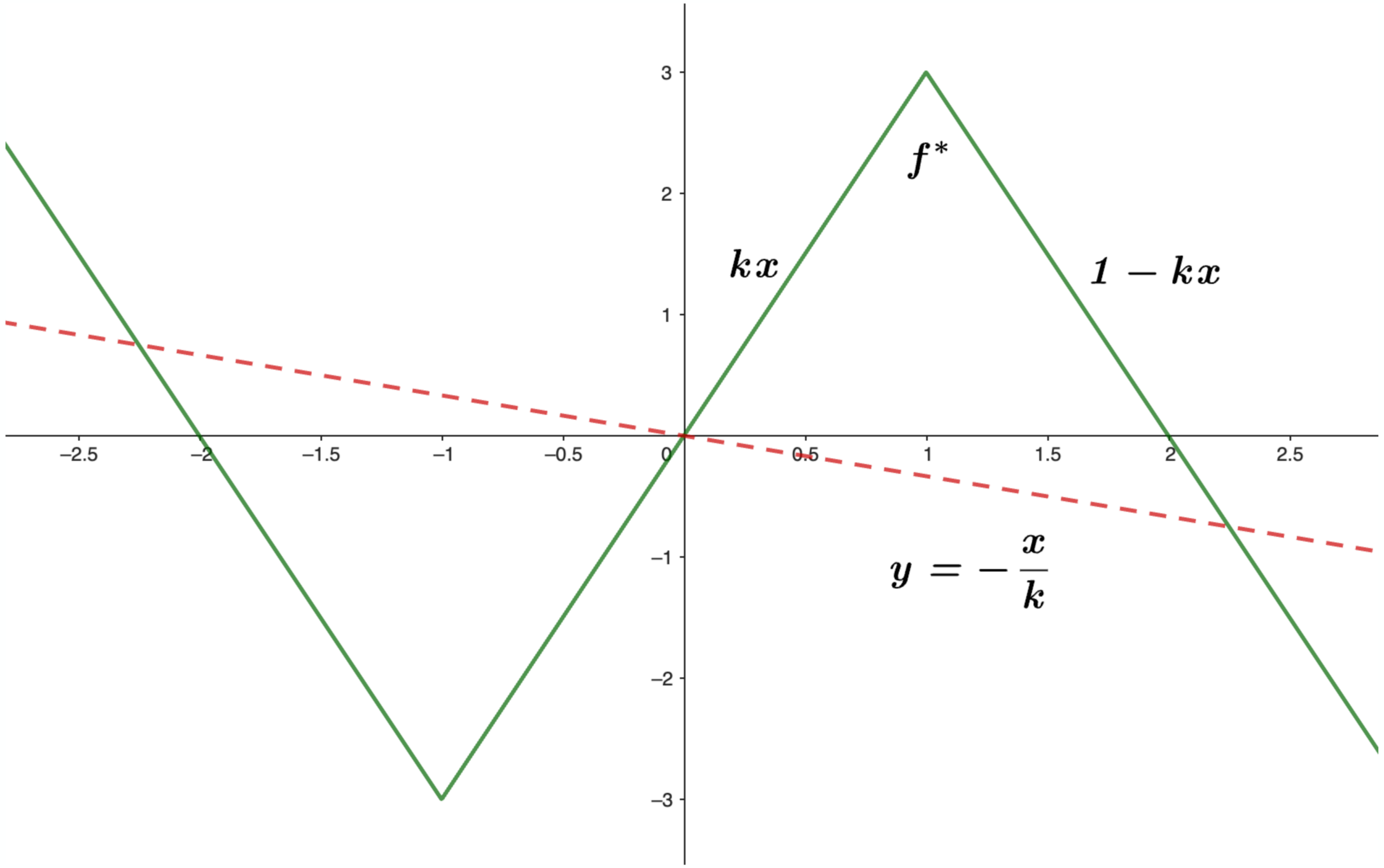
$$\partial_t f_x(t,x^*) = -\partial_x f_x(t,x^*) \cdot \int_{\mathbb{R}} \frac{y-x^*}{(f(t,y)-f(t,x^*))^2+(y-x^*)^2}\,\mathrm{d}y$$
$$-2\int_{\mathbb{R}} \frac{f_x(t,x^*)-l(t,y,x^*)}{(y-x^*)^2} \cdot \frac{1+f_x(t,x^*)l(t,y,x^*)}{\left(1+l^2(t,y,x^*)\right)^2}\,\mathrm{d}y,$$

where $l(t,y,x^*) := \frac{f(t,y)-f(t,x^*)}{y-x^*}$ is the slope between $(x^*, f(t,x^*))$ and $(y, f(t,y))$.

- $f_x(t,x^*) - l(t,y,x^*) > 0,$ and $1 + f_x(t,x^*)l(t,y,x^*) > 0$ if $l(t,y,x^*) > -\frac{1}{f_x(t,x^*)}$.

- Integrand < 0 only if $l(t,y,x^*) < -\frac{1}{f_x(t,x^*)}$.

设想中的最坏情况



conjectured worst case scenario

- $l(t, y, 0)$ reaches $-\frac{1}{f_x(t,0)}$ as fast as possible in the conjectured worst case.
- $\frac{d}{dt}\|f_x(t,\cdot)\|_{L^\infty} \approx -\frac{k(3-k^2)}{(1+k^2)^2} \leq 0$, provided that $k = \|f_x\|_{L^\infty} \leq \sqrt{3}$.
- Maximum principle holds when $\|f_0'\| \lesssim \sqrt{3}$?

The Key Idea of the Proof

极值原理的证明:命题的转化

- Define

$$\mathscr{F}_k := \{f \in \mathbb{R}^{[0,+\infty)} : \text{ piecewise differentiable}, f_x(0) = k, -k \le f_x(x) \le k\},$$

$$H_k(l) := \frac{2(k-l)(1+kl)}{(1+l^2)^2},$$

$$I_{[f]} := \int_0^\infty \frac{2}{y^2} \cdot \frac{\left(f_x(0) - l_{[f]}(y)\right) \cdot \left(1 + f_x(0) l_{[f]}(y)\right)}{\left(1 + l_{[f]}^2(y)\right)^2} \, \mathrm{d}y = \int_0^\infty \frac{1}{y^2} H_k(l_{[f]}(y)) \, \mathrm{d}y,$$

where $l_{[f]}(y) := \frac{f(y) - f(0)}{y}$.

- Need to prove: If $k \le L$, $I_{[f]} \ge 0$ for any $f \in \mathscr{F}_k$.
- Problem of calculus of variations. What is the minimizer of the functional $I_{[f]}$?

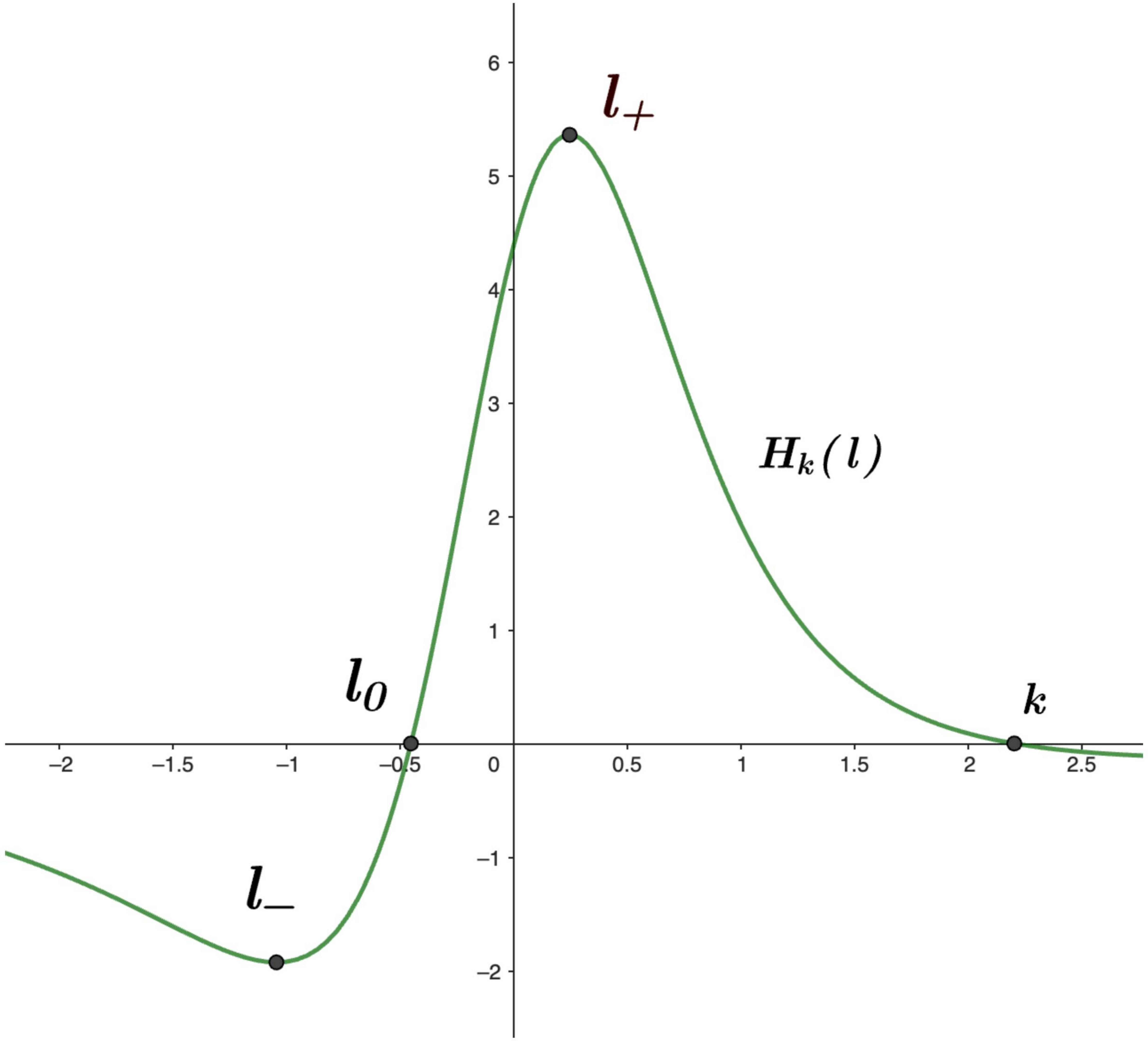
极值原理的证明:积分核的单调性

- $I_{[f]} = \int_{\mathbb{R}} \frac{1}{y^2} H_k(l_{[f]}(y))\, \mathrm{d}y$, where

  $l_{[f]}(y)$ is the slope and $k = \|f_x\|_{L^\infty}$.

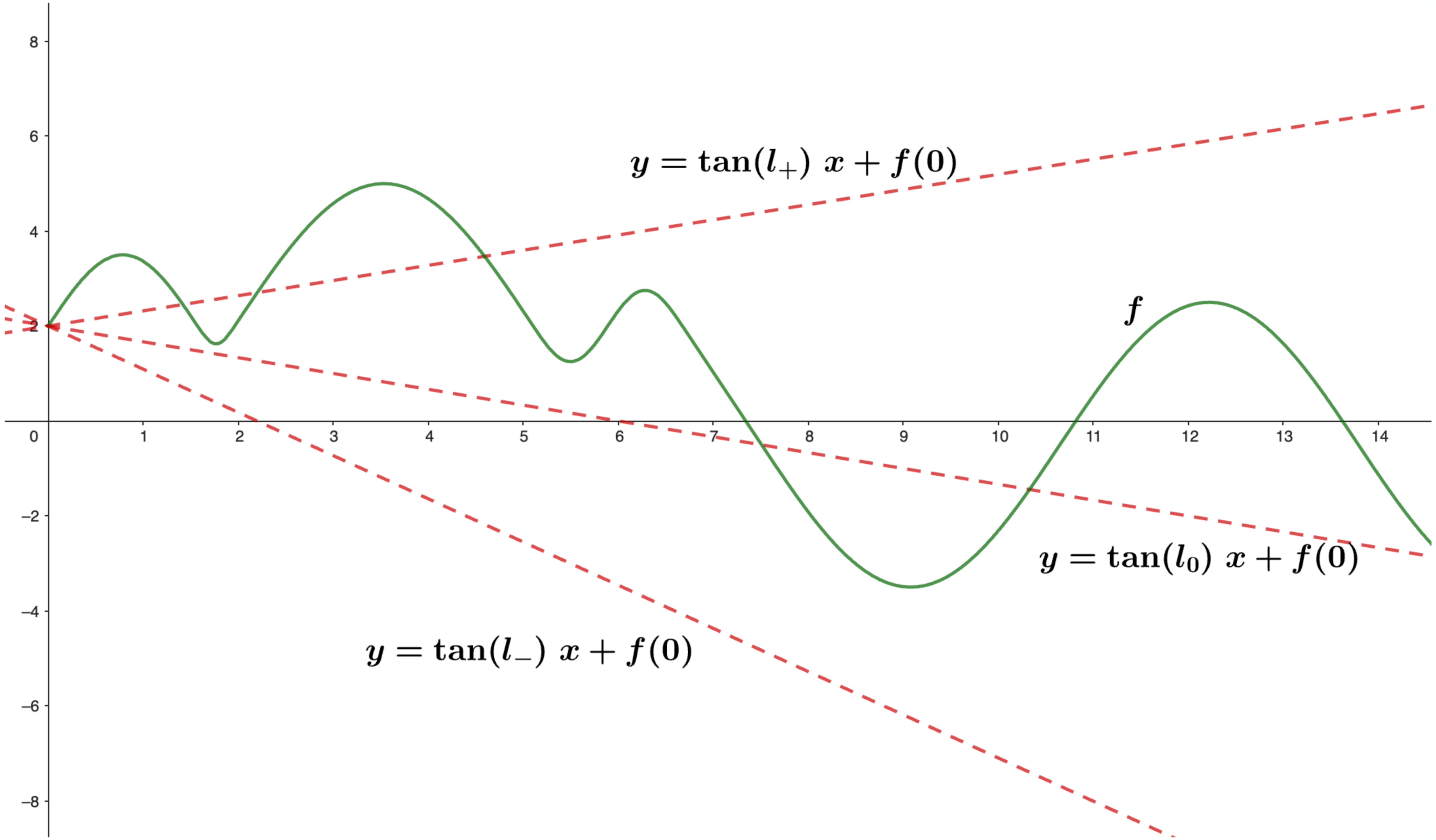
- $H_k(l) := \dfrac{2(k-l)(1+kl)}{(1+l^2)^2}$.

$$\begin{cases} H_k(l) \text{ decreases}, & l \in (-\infty, l_-], \\ H_k(l) \text{ increases}, & l \in [l_-, l_+], \\ H_k(l) \text{ decreases}, & \beta \in [l_+, k], \\ H_k(l_-) = \min\limits_{-\infty \le l \le k} H_k(l), & \\ H_k(l) \ge 0, & \beta \in [l_0, k], \\ H_k(l) \le 0, & \beta \in (-\infty, l_0]. \end{cases}$$



极值原理的证明: 最小化过程

- $I_{[f]} = \int_{\mathbb{R}} \frac{1}{y^2} H_k(l_{[f]}(y))\,\mathrm{d}y$, where $l_{[f]}(y)$ is the slope between $(y, f(y))$ and $(0, f(0))$, and $k = \|f_x\|_{L^\infty}$.

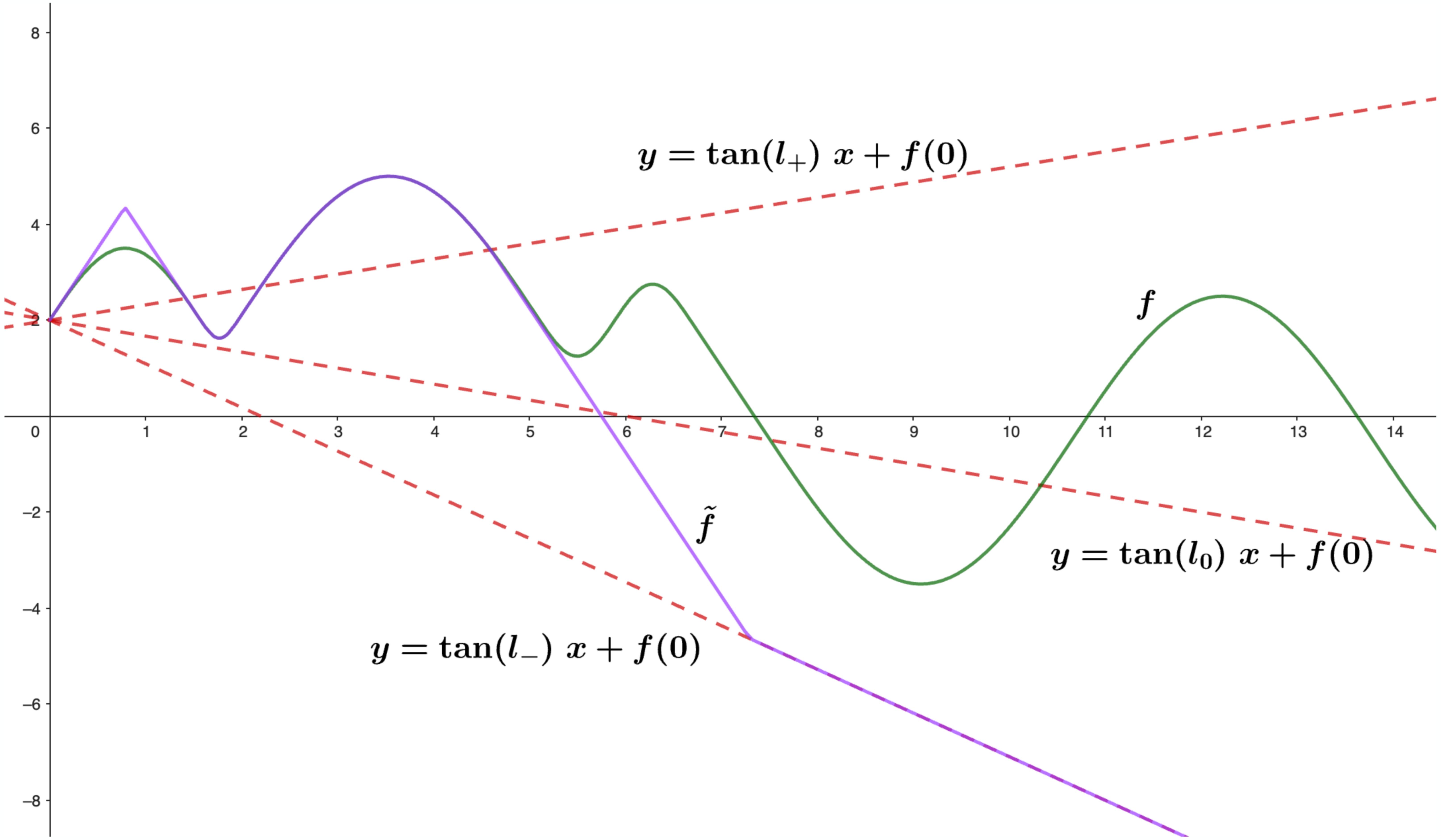


极值原理的证明: 最小化过程

- $I_{[f]} = \int_{\mathbb{R}} \frac{1}{y^2} H_k(l_{[f]}(y))\, \mathrm{d}y,$ where $l_{[f]}(y)$ is the slope between $(y, f(y))$ and $(0, f(0))$, and $k = \|f_x\|_{L^\infty}$.

- $H_k(l_{[\tilde{f}]}(y)) \leq H_k(l_{[f]}(y))$.

- $I_{[\tilde{f}]} \leq I_{[f]}$.

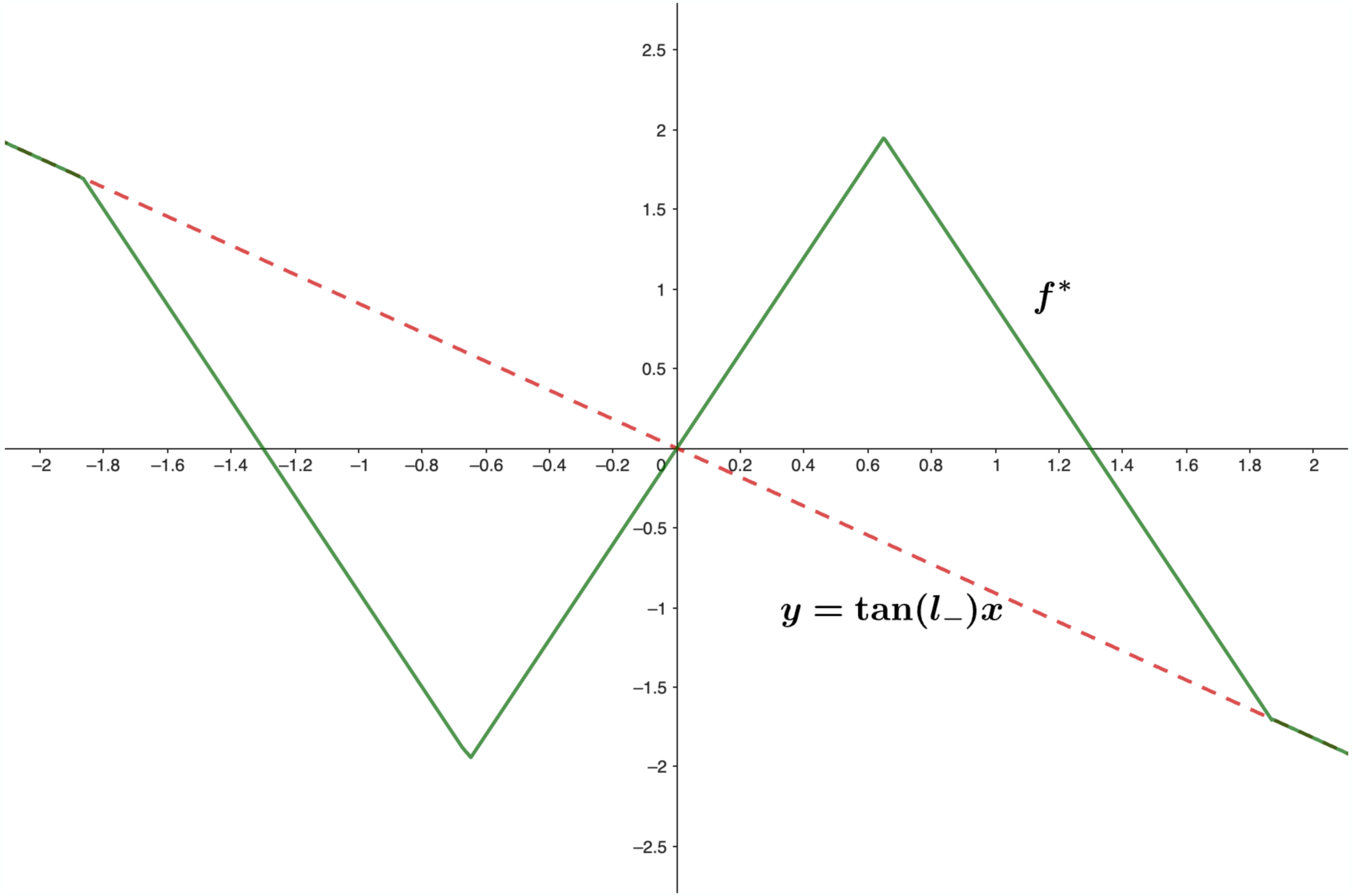


极值原理的证明: 最小化子的计算

$$I_{[\tilde{f}]} \geq \begin{cases} \frac{1}{x_+(k+\tan(\frac{2\alpha}{3}-\frac{\pi}{6}))}\left[1-F\left(\tan(\frac{2\alpha}{3}-\frac{\pi}{6})\right)\right] \\ \qquad +\frac{1}{x_2(k+\tan(\frac{2\alpha}{3}-\frac{\pi}{6}))}\left[F\left(\tan(\frac{2\alpha}{3}-\frac{\pi}{6})\right)-F(-k)\right], & \alpha \leq \frac{3\pi}{10}, \\ \frac{1}{x_+(k+\tan(\frac{2\alpha}{3}-\frac{\pi}{6}))}\left[1-F\left(\tan(\frac{2\alpha}{3}-\frac{\pi}{6})\right)\right] \\ \qquad +\frac{1}{x_2(k+\tan(\frac{2\alpha}{3}-\frac{\pi}{6}))}\left[F\left(\tan(\frac{2\alpha}{3}-\frac{\pi}{6})\right)-F\left(\tan(\frac{2\alpha}{3}-\frac{\pi}{2})\right)\right] \\ \qquad +\frac{k+\tan(\frac{2\alpha}{3}-\frac{\pi}{2})}{x_2(k+\tan(\frac{2\alpha}{3}-\frac{\pi}{6}))}H_k(\tan(\frac{2\alpha}{3}-\frac{\pi}{2})), & \alpha > \frac{3\pi}{10}, \end{cases}$$

where $\alpha := \arctan k$ and $F(l) := \frac{2(k-l)(1+kl)}{1+l^2} \Longrightarrow I_{[\tilde{f}]} \geq 0$ if $k = \|f_0'\|_{L^\infty} \leq L$.

极值原理的证明: L的最优性



worst case scenario

衰减性的证明

Define $M(t) := \max_{x\in\mathbb{R}} |f_x(t,x)|$, then

$$
\begin{aligned}
M'(t) &= -\int_{\mathbb{R}} \frac{2}{(y-x_t)^2} \cdot \frac{\left(f_x(t,x_t) - l(t,y,x_t)\right)\cdot\left(1+f_x(t,x_t)l(t,y,x_t)\right)}{\left(1+l^2(t,y,x_t)\right)^2}\,\mathrm{d}y \\
&= -\int_{\mathbb{R}} \frac{2}{(y-x_t)^2} \frac{\varepsilon\left(f_x(t,x_t) - l(t,y,x_t)\right)}{\left(1+l^2(t,y,x_t)\right)^2}\,\mathrm{d}y \\
&\quad -\int_{\mathbb{R}} \frac{2}{(y-x_t)^2} \cdot \frac{\left(f_x(t,x_t) - l(t,y,x_t)\right)\cdot\left(1-\varepsilon+f_x(t,x_t)l(t,y,x_t)\right)}{\left(1+l^2(t,y,x_t)\right)^2}\,\mathrm{d}y \\
&=: -\varepsilon I_1 - I_{2,\varepsilon},
\end{aligned}
$$

where x_t is a maximum point.

衰减性的证明

- $M(t) := \max_{x\in\mathbb{R}} |f_x(t,x)|$,

$$M'(t) = -\int_{\mathbb{R}} \frac{2}{(y-x_t)^2} \frac{\varepsilon\left(f_x(t,x_t) - l(t,y,x_t)\right)}{\left(1 + l^2(t,y,x_t)\right)^2} \,\mathrm{d}y$$

$$-\int_{\mathbb{R}} \frac{2}{(y-x_t)^2} \cdot \frac{\left(f_x(t,x_t) - l(t,y,x_t)\right) \cdot \left(1 - \varepsilon + f_x(t,x_t) l(t,y,x_t)\right)}{\left(1 + l^2(t,y,x_t)\right)^2} \,\mathrm{d}y$$

$$=: -\varepsilon I_1 - I_{2,\varepsilon}\,.$$

- $I_1 \approx C(-\Delta)^{\frac{1}{2}} f_x(t,x_t)$
- $I_{2,\varepsilon}$ is a perturbation of $I_{[f]}$, thus $I_{2,\varepsilon} \geq 0$ for some small enough $\varepsilon > 0$.
- $M'(t) \leq -\varepsilon C(-\Delta)^{\frac{1}{2}} f_x(t,x)$.

Thanks