Fourier–Mukai Transforms and Vector Bundles on Elliptic Curves

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Vector bundles on elliptic curves

- k: algebraically closed field
- C: elliptic curve over k with identity $e \in C(k)$
- For $x \in C(k)$, let $\mathcal{P}_x = \mathcal{O}(x - e)$
- Letting $x \mapsto \mathcal{P}_x$ gives an isomorphism $C(k) \cong \mathrm{Pic}^0(C)$

Question

What about vector bundles of higher rank?

Atiyah's classification

Theorem (Atiyah '57)

Let $\mathcal{E}(r,d)$ be the set of indecomposable vector bundles on C of rank r and degree d.
(1) There is a canonical bijection $C(k) \cong \mathcal{E}(r,d)$, $x \mapsto \mathcal{E}_{r,d,x}$ such that $\mathcal{E}_{1,0,x} = \mathcal{P}_x$.
(2) If $\gcd(r,d) = 1$, then $\mathcal{E}_{r,d,x}$ is simple. Moreover, for any $n \in \mathbb{Z}_+$, $\mathcal{E}_{nr,nd,x}$ is an iterated extension of $\mathcal{E}_{r,d,x}$.
(3) Suppose k has characteristic 0, and write $\mathcal{E}_n = \mathcal{E}_{n,0,e}$. Then

$$\mathcal{E}_n \otimes \mathcal{E}_m = \bigoplus_{k=1}^{\min\{m,n\}} \mathcal{E}_{m+n+1-2k}.$$

The method of G. Hein and D. Ploog

In 2005, G. Hein and D. Ploog give a new proof of Atiyah's theorem with the help of two ideas:

- Fourier–Mukai transforms
- Slope stability

Fourier–Mukai tranforms

The general mechanism of Fourier–Mukai transforms is the following. We denote by $\mathrm{D}^{\mathrm{b}}(X)$ the bounded derived category of coherent sheaves of a variety X.

Definition

Let X, Y be smooth projective varieties and $\mathcal{K} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$. The Fourier–Mukai transform with kernel $\mathcal{K}$ is the functor

$$\mathrm{FM}_{\mathcal{K}} : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y), \quad \mathcal{F} \mapsto Rp_{Y*}(p_X^* \mathcal{F} \otimes^L \mathcal{K}),$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are canonical projections.

Fourier–Mukai transforms

Take $X = Y = C$ our elliptic curve, and $\mathcal{F} = \mathcal{P} \in \mathrm{Pic}^0(C \times C)$ be the Poincaré bundle: get an endofunctor $\mathrm{FM} = \mathrm{FM}_{\mathcal{P}}$ of $\mathrm{D^b}(C)$.

Proposition

(1) FM is an equivalence, and $\mathrm{FM}^2 = \iota^*[-1]$, where $\iota : C \to C$ is the inverse map.
(2) For $x \in C(k)$, let k_x be the skyscraper sheaf at x. Then $\mathrm{FM}(k_x) = \mathcal{P}_x$ and $\mathrm{FM}(\mathcal{P}_x) = k_{-x}[-1]$.

Problem

If $\mathcal{E}$ is a vector bundle, then $\mathrm{FM}(\mathcal{E}) \in \mathrm{D^b}(C)$ is not a vector bundle in general: it might not even concentrate in one degree.

Luckily, things get good if we assume $\mathcal{E}$ to be semistable.

Slope stability

Let $\mathcal{E}$ be a vector bundle on C with rank r and degree d. The slope of $\mathcal{E}$ is $\mu(\mathcal{E}) = d/r \in \mathbb{Q}$.

Definition

$\mathcal{E}$ is stable (semistable resp.) if for any nontrivial subbundle $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$ ($\mu(\mathcal{F}) \leq \mu(\mathcal{E})$, resp.).

Proposition

Indecomposable vector bundles on elliptic curves are semistable.

We don't lose anything when restricting to semistable vector bundles.

Fourier–Mukai transform of semistable vector bundles

For $\mu \in \mathbb{Q}$, let $\mathrm{Vect}^{\mathrm{ss}}(C)_\mu$ be the category of vector bundles of slope μ.
Let $\mathrm{Vect}^{\mathrm{ss}}(C)_\infty = \mathrm{Coh}(C)_{\mathrm{tor}}$ be the category of torsion coherent sheaves.

Theorem

Let $\mu \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. Let $\mathcal{E} \in \mathrm{Vect}^{\mathrm{ss}}(C)_\mu$, then

$$\mathrm{FM}(\mathcal{E}) \in \begin{cases} \mathrm{Vect}^{\mathrm{ss}}(C)_{-\mu^{-1}}, & \mu \in \mathbb{Q}_{>0} \cup \{\infty\} \\ \mathrm{Vect}^{\mathrm{ss}}(C)_{-\mu^{-1}}[-1], & \mu \in \mathbb{Q}_{\leq 0}. \end{cases}$$

Slope translation

Last theorem: FM restricts to an equivalence (modulo shifts)

$$\mathrm{Vect}^{\mathrm{ss}}(C)_\mu \simeq \mathrm{Vect}^{\mathrm{ss}}(C)_{-\mu^{-1}}.$$

On the other hand, there is an equivalence

$$\mathrm{Vect}^{\mathrm{ss}}(C)_\mu \simeq \mathrm{Vect}^{\mathrm{ss}}(C)_{\mu+1}, \quad \mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}(e).$$

$\mu \mapsto -\mu^{-1}$ and $\mu \mapsto \mu + 1$ generates the transitive action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$! By checking relations, we obtain:

Corollary

There are canonical equivalences between all $\mathrm{Vect}^{\mathrm{ss}}(C)_\mu$.

This makes clear the structure of $\mathrm{Vect}^{\mathrm{ss}}(C)_\mu$, since $\mathrm{Coh}(C)_{\mathrm{tor}}$ is easy.

Convolution products

Now we turn to multiplicative structures.

Definition

Let $p_1, p_2, m : C^2 \to C$ be the two projections and the multiplication map, respectively. For $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(C)$, their convolution product is

$$\mathcal{F} \star \mathcal{G} = Rm_*(p_1^*\mathcal{F} \otimes^L p_2^*\mathcal{G}).$$

Proposition

$$\mathrm{FM}(\mathcal{F} \star \mathcal{G}) = \mathrm{FM}(\mathcal{F}) \otimes^L \mathrm{FM}(\mathcal{G}).$$

Calculation of convolution products

Let $\mathcal{F}_n = \mathcal{O}_e/\mathfrak{m}_e^n$ be the length n indecomposable torsion sheaf at e, so that $\mathrm{FM}(\mathcal{F}_n) = \mathcal{E}_n = \mathcal{E}_{n,0,e}$. By the last proposition, (3) in Atiyah's theorem is equivalent to

$$\mathcal{F}_n \star \mathcal{F}_m = \bigoplus_{k=1}^{\min\{m,n\}} \mathcal{F}_{m+n+1-2k}$$

To calculate $\mathcal{F}_n \star \mathcal{F}_m$ we can restrict to the formal neighborhood of e.

- $\hat{\mathcal{O}}_e$ inherits a formal group structure $\Delta : \hat{\mathcal{O}}_e \to \hat{\mathcal{O}}_e \hat{\otimes} \hat{\mathcal{O}}_e$
- $\mathcal{F}_n$ is the $\hat{\mathcal{O}}_e$-module $\hat{\mathcal{O}}_e/\hat{\mathfrak{m}}_e^n$
- $p_1^*\mathcal{F}_n \otimes^L p_2^*\mathcal{F}_m$ is the $\hat{\mathcal{O}}_e \hat{\otimes} \hat{\mathcal{O}}_e$-module $\hat{\mathcal{O}}_e/\hat{\mathfrak{m}}_e^n \otimes \hat{\mathcal{O}}_e/\hat{\mathfrak{m}}_e^m$
- $\mathcal{F}_n \star \mathcal{F}_m$ is $\hat{\mathcal{O}}_e/\hat{\mathfrak{m}}_e^n \otimes \hat{\mathcal{O}}_e/\hat{\mathfrak{m}}_e^m$ with $\hat{\mathcal{O}}_e$-module structure induced by Δ

Formal group laws

Choosing a coordinate $\hat{\mathcal{O}}_e \cong k[[x]]$, we arrive at the following question:

Question

Given a formal group law $F(x, y) \in k[[x, y]]$, what is $k[x, y]/(x^n, y^m)$ as a $k[F(x, y)]$-module?

Theorem

(1) Every formal group law in characteristic 0 is isomorphic to the additive formal group law $F(x, y) = x + y$.
(2) (Lazard '55) Formal group laws over algebraically closed fields of characteristic p is classified by their heights, which take value in $\mathbb{Z}_+ \cup \{\infty\}$.

Characteristic 0 case

In characteristic 0, we want to know the structure of $k[x, y]/(x^n, y^m)$ as a $k[x + y]$-module. Write n instead of $\mathcal{F}_n$.

Method 1:

- Direct calculation shows $2 \star n = (n - 1) \oplus (n + 1)$
- Using $(2 \star n) \star m = 2 \star (n \star m)$, induction gives the desired formula for $n \star m$

Method 2 (thanks to inspections of Liang Xiao and Kaiyuan Gu):

- Regard $k[x]/(x^n)$ as an irreducible $\mathfrak{sl}_2$-representation, so that the raising operator acts as multiplication by x
- In the tensor product representation $k[x, y]/(x^n, y^m)$, the raising operator acts as multiplication by $x + y$
- Use the tensor product formula of $\mathfrak{sl}_2$-representation

Characteristic p case

Algorithm (Conjectured):

1. The answer does NOT depend on the formal group
2. If $m \leq p^k$ and $m \star n = \bigoplus_{i=1}^{m} a_i$ (a_i can be 0), then $m \star (n + p^k) = \bigoplus_{i=1}^{m} (a_i + p^k)$
3. If $m, n \leq p^k$ and $m \star n = \bigoplus_{i=1}^{m} a_i$ (a_i can be 0), then $m \star (p^k - n) = \bigoplus_{i=1}^{m} (p^k - a_i)$
4. Step 2 implies: if $m \leq n \leq p^k$ and $m + n > p^k$, then $m \star n = a_1 \oplus \cdots \oplus a_{p^k - n} \oplus (p^k)^{\oplus (m+n-p^k)}$ $(0 < a_1, \ldots, a_{p^k-n} < p^k)$
5. By step 1,2 may assume $p^k < m \leq n \leq \frac{p^{k+1}}{2}$
6. If $k = 0$ then $m \star n = \bigoplus_{i=1}^{\min\{m,n\}} (m + n + 1 - 2i)$
7. Write $m = q \cdot p^k + m_0$ and $n = q' \cdot p^k + n_0$, with $0 \leq m_0, n_0 < p^k$

Characteristic p case

If $m_0 \leq n_0$, write $m_0 \star n_0 = \left(\bigoplus_{i=1}^{m_0-r} a_i\right) \oplus (p^k)^{\oplus r}$ $(0 < a_1, \ldots, a_{m_0-r} < p^k$, so that $r = \max\{0, m_0 + n_0 - p^k\})$, then

$$\begin{aligned} m \star n = & \bigoplus_{j=0}^{q} \bigoplus_{i=1}^{m_0-r} \left(a_i + (2j + q' - q)p^k\right) \\ & \oplus \bigoplus_{j=0}^{q-1} \left((2j + q' - q + 1)p^k\right)^{\oplus |p^k - m_0 - n_0|} \oplus \left((q' + q + 1)p^k\right)^{\oplus r} \\ & \oplus \bigoplus_{j=0}^{q-1} \bigoplus_{i=1}^{m_0-r} \left((2j + q' - q + 2)p^k - a_i\right) \\ & \oplus \bigoplus_{j=0}^{q-1} \left((2j + q' - q + 2)p^k\right)^{\oplus (n_0 - m_0)} \end{aligned}$$

Characteristic p case

If $m_0 > n_0$, write $m_0 \star n_0 = \left(\bigoplus_{i=1}^{n_0-r} a_i\right) \oplus (p^k)^{\oplus r}$ ($0 < a_1, \dots, a_{n_0-r} < p^k$, so that $r = \max\{0, m_0 + n_0 - p^k\}$), then

$$
\begin{aligned}
m \star n = & \bigoplus_{j=0}^{q} \left((2j + q' - q)p^k\right)^{\oplus(m_0-n_0)} \\
& \oplus \bigoplus_{j=0}^{q} \bigoplus_{i=1}^{n_0-r} \left(a_i + (2j + q' - q)p^k\right) \\
& \oplus \bigoplus_{j=0}^{q-1} \left((2j + q' - q + 1)p^k\right)^{\oplus|p^k-m_0-n_0|} \oplus \left((q' + q + 1)p^k\right)^{\oplus r} \\
& \oplus \bigoplus_{j=0}^{q-1} \bigoplus_{i=1}^{n_0-r} \left((2j + q' - q + 2)p^k - a_i\right)
\end{aligned}
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Thank you!